

# UNIQUENESS RESULTS FOR ILL POSED CHARACTERISTIC PROBLEMS IN CURVED SPACE-TIMES

ALEXANDRU D. IONESCU AND SERGIU KLAINERMAN

**ABSTRACT.** We prove two uniqueness theorems concerning linear wave equations; the first theorem is in Minkowski space-times, while the second is in the domain of outer communication of a Kerr black hole. Both theorems concern ill-posed Cauchy problems on bifurcate, characteristic hypersurfaces. In the case of the Kerr space-time, the hypersurface is precisely the event horizon of the black hole. The uniqueness theorem in this case, based on two Carleman estimates, is intimately connected to our strategy to prove uniqueness of the Kerr black holes among smooth, stationary solutions of the Einstein-vacuum equations, as formulated in [14].

## CONTENTS

1. Introduction	1
1.1. A model problem in Minkowski spaces	4
1.2. The main theorem in the Kerr spaces	4
2. Unique continuation and conditional Carleman inequalities	5
2.1. General considerations	5
2.2. A conditional Carleman inequality of sufficient generality	6
3. Proof of Theorem 1.2	8
3.1. The first Carleman inequality in Kerr spaces	8
3.2. The second Carleman inequality in Kerr spaces	12
3.3. Vanishing of the tensor $\mathcal{S}$	14
4. Proof of Theorem 1.1	20
Appendix A. Explicit computations in the Kerr spaces	24
References	27

## 1. INTRODUCTION

The goal of the paper is to prove two uniqueness results for the Cauchy problem in the exterior of a bifurcate characteristic surface. In the simplest case of the wave equation in Minkowski space  $\mathbb{R}^{1+d}$ ,

$$\square\phi = 0, \quad \square = -\partial_t^2 + \sum_{i=1}^d \partial_i^2$$

the problem is to find solutions in the exterior domain  $\mathcal{E}_a = \{(t, x) : |x| > |t| + a\}$ ,  $a \geq 0$ , with prescribed data on the boundary  $\mathcal{H}_a = \{(t, x) : |t| = |x| + a\}$ . The problem is known to be *ill posed*, that is,

- (1) Solutions may not exist for smooth, non-analytic, initial conditions.
- (2) There is no continuous dependence on the data.

The situation is similar to the better known case of the Cauchy problem prescribed on a time-like characteristic hypersurface, such as  $x^d = 0$ . The Cauchy–Kowalewski theorem allows one to solve the problem for analytic initial data, but solutions may not exist in the smooth case. It is known in fact that smooth solutions cannot be prescribed freely, since certain necessary compatibilities may be violated.

Though existence fails, one can often prove uniqueness. A general result due to Holmgren, improved by F. John [8], shows that the non-characteristic initial value problem for linear equations with analytic coefficients is locally unique in the class of smooth solutions, see [9]. The case of equations with smooth coefficients is considerably more complicated. An important counterexample to uniqueness was provided by P. Cohen [5], see also [12] and [1] for more general families of examples. Thus, in the case of the Cauchy problem for a time-like hypersurface (such as  $x^d = 0$ ), even a zero order, smooth, perturbation of the wave operator  $\square$  can cause uniqueness to fail. We note also that, there cannot be, in general (unless one considers solutions with suitable decay at infinity such as discussed in [16]), unique continuation across characteristic hyperplanes, see the counterexample and the discussion in [13, Theorem 8.6.7]. On the other hand, there exist conditions which can guarantee uniqueness, most importantly those of Hörmander [13, Chapter 28]. See also [19], [21] and the references therein for uniqueness results under partial analyticity assumptions. These results prove uniqueness for a large class of problems which include, in particular, the Cauchy problem on an arbitrary, non-characteristic, time-like hypersurface for the wave equation  $\square_{\mathbf{g}}\phi = 0$ , corresponding to a time independent Lorentz metric of the form  $-\mathbf{g}_{00}(x)dt^2 + \mathbf{g}_{ij}(x)dx^i dx^j$  with  $\mathbf{g}_{00} > 0$  and  $(\mathbf{g}_{ij})_{i,j=1}^d$  positive definite. The method of proof for these and other modern unique continuation results is based on Carleman type estimates.

The case of ill posed problems for bifurcate characteristic hypersurfaces, i.e. surfaces composed of two characteristic hypersurfaces which intersect transversally, seems to have been first studied by Friedlander<sup>1</sup> [6], by using a variation of Holmgren’s method of proof. The same problem for equations with smooth coefficients, seems not to have been specifically considered in the literature. Yet it is precisely this case which seems to be of considerable importance in General Relativity, particularly for the problem of uniqueness of stationary, smooth solutions of the Einstein field equations, see discussion in [14]. Indeed, it turns out that remarkable simplifications occur for the geometry of bifurcate horizons for general, stationary, asymptotically flat black hole solutions of the Einstein-vacuum

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<sup>1</sup>In [7] he also considers a similar, ill posed, characteristic problem at infinity, concerning uniqueness of solutions with identical radiation fields.

equations, verifying reasonable regularity assumptions. For such regular black hole space-times, Hawking has shown, see [10], then there must exist an additional Killing vector-field defined on the event horizon, tangent to the generators of the horizon. In the case when the space-time is real analytic one can extend this additional Killing vector-field to the entire exterior region, and deduce that the space-time must be not only stationary but also axially symmetric. A satisfactory uniqueness result (due to Carter [3] and Robinson [20]) is known for stationary solutions which have this additional symmetry. However, in the smooth, non-analytic case, the problem of extending Hawking's Killing vector-field from the horizon to the exterior region leads to an ill posed characteristic problem. This appears to be the key obstruction to proving the analogue of Hawking's uniqueness theorem in the class of smooth, non-analytic space-times.

Motivated by this latter problem, to avoid the analyticity assumption we are proposing a completely different approach<sup>2</sup> based on the following ingredients.

- (1) The Kerr space-times can be locally characterized, among stationary solutions, by the vanishing of a four covariant tensor-field, called the Mars-Simon tensor  $\mathcal{S}$  introduced in [17].
- (2) The Mars-Simon tensor-field  $\mathcal{S}$  verifies a covariant system of wave equation of the form (see also first equation in (1.6)) ,

$$\square_{\mathbf{g}}\mathcal{S} = \mathcal{A} \cdot \mathbf{D}\mathcal{S} + \mathcal{B} \cdot \mathcal{S}. \quad (1.1)$$

Moreover, since  $\mathbf{g}$  is stationary, we know that there exists a globally defined Killing vector-field  $\xi$ , which is time-like at space-like infinity. Thus it is easy to verify that the Lie derivative of  $\mathcal{S}$  with respect to  $\xi$  vanishes identically.

$$\mathcal{L}_{\xi}\mathcal{S} = 0. \quad (1.2)$$

- (3) One can show that the tensor-field  $\mathcal{S}$  vanishes identically on the bifurcate horizon  $\mathcal{H}$  of the stationary metric  $g$ . We show this by making an assumption (automatically satisfied on a Kerr metric) concerning the vanishing of a complex scalar on the bifurcate sphere of the horizon.
- (4) Using a first Carleman estimate for the covariant wave equation (1.1) we show that  $\mathcal{S}$  vanishes in a neighborhood of the bifurcate sphere. This step does not require condition (1.2), indeed it is a result that applies to general equation of type (1.1) in a neighborhood of a regular bifurcate characteristic hypersurface, for a general Lorentz metric  $\mathbf{g}$ .
- (5) To extend the vanishing of  $\mathcal{S}$  to the entire domain of outer communication we need a more sophisticated Carleman estimate which depends in an essential fashion, among other considerations, on equation (1.2).

In this paper we prove, see Theorem 1.2, a global uniqueness result for tensor-field solutions to covariant equations of the form (1.1) and (1.2) on the domain of outer communication of a Kerr background, which vanish on the event horizon. The condition (1.2)

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<sup>2</sup>See the longer discussion in [14].

relative to the stationary Killing vector-field  $\xi$ , which is important to prove a global result, is justified by the fact that the problem of uniqueness of Kerr is restricted, naturally, to stationary solutions of the Einstein vacuum equations (see discussion in [14]). We also discuss a simple model problem, see Theorem 1.1, concerning scalar linear wave equations in the exterior domain  $\mathcal{E} = \mathcal{E}_1$  of the Minkowski space-time with prescribed data on the characteristic boundary  $\mathcal{H} = \mathcal{H}_1$ .

We would like to thank A. Rendall for bringing to our attention the work of Friedlander, [6], [7].

**1.1. A model problem in Minkowski spaces.** Assume  $d \geq 1$  and let  $(\mathcal{M} = \mathbb{R} \times \mathbb{R}^d, \mathbf{m})$  denote the usual Minkowski space of dimension  $d + 1$ . We define the subsets of  $\mathcal{M}$

$$\mathcal{E} = \{(t, x) \in \mathcal{M} : |x| > |t| + 1\}, \quad (1.3)$$

and

$$\mathcal{H} = \delta(\mathcal{E}) = \{(t, x) \in \mathcal{M} : |x| = |t| + 1\}. \quad (1.4)$$

Let  $\bar{\mathcal{E}} = \mathcal{E} \cup \mathcal{H}$ . Our first theorem concerns a uniqueness property of solutions of wave equations on  $\mathcal{E}$ .

**Theorem 1.1.** *Assume  $\phi \in C^2(\mathcal{M})$ ,  $A, B^l \in C^0(\mathcal{M})$ ,  $l = 0, \dots, d$ , and*

$$\square\phi = A \cdot \phi + \sum_{l=0}^d B^l \cdot \partial_l \phi \quad \text{on } \mathcal{E}. \quad (1.5)$$

*Assume that  $\phi \equiv 0$  on  $\mathcal{H}$ . Then  $\phi \equiv 0$  on  $\bar{\mathcal{E}}$ .*

Theorem 1.1 extends easily to diagonal systems of scalar equations. We remark that in Theorem 1.1 we do not assume any global bounds on the coefficients  $A$  and  $B^l$ . Also, we make no assumption on the vanishing of the derivatives of  $\phi$  on  $\mathcal{H}$ , which is somewhat surprising given that  $\square$  is a second order operator. This is possible because of the special bifurcate characteristic structure of the surface  $\mathcal{H}$ .

The proof of Theorem 1.1, which is given in section 4, follows from a standard Carleman inequality with a suitably defined pseudo-convex weight. However, the simple statement of Theorem 1.1 appears to be new. We include it here mostly as a model result to illustrate, in a very simple case, the connection between bifurcate characteristic horizons and unique continuation properties of solutions of wave equations.

**1.2. The main theorem in the Kerr spaces.** Let  $(\mathbf{K}^4, \mathbf{g})$  denote the maximally extended Kerr spacetime of mass  $m$  and angular momentum  $ma$  (see the appendix for some details and explicit formulas). We assume

$$m > 0 \text{ and } a \in [0, m).$$

Let  $\mathbf{E}^4$  denote a domain of outer communication of  $\mathbf{K}^4$ , and  $\mathcal{H} = \delta(\mathbf{E}^4)$  the corresponding event horizon. Let  $\mathbf{M}^4$  denote an open neighborhood of  $\mathbf{E}^4 \cup \mathcal{H}$  in  $\mathbf{K}^4$ , and let  $\xi$  denote a Killing vector field on  $\mathbf{E}^4$  which is timelike at the spacelike infinity in  $\mathbf{E}^4$ . Let  $\mathbb{T}(\mathbf{M}^4)$  denote the space of smooth vector-fields on  $\mathbf{M}^4$ , and let  $\mathbb{T}_s^r(\mathbf{M}^4)$ ,  $r, s \in \mathbb{Z}_+$ , denote the

space of complex-valued tensor-fields of type  $(r, s)$  on  $\mathbf{M}^4$ . Our main theorem concerns a uniqueness property of certain solutions of covariant wave equations on  $\mathbf{E}^4$ .

**Theorem 1.2.** *Assume  $k \in \mathbb{Z}_+$ ,  $\mathcal{S} \in \mathbb{T}_k^0(\mathbf{M}^4)$ ,  $\mathcal{A} \in \mathbb{T}_k^k(\mathbf{M}^4)$ ,  $\mathcal{B} \in \mathbb{T}_k^{k+1}(\mathbf{M}^4)$ ,  $\mathcal{C} \in \mathbb{T}_k^k(\mathbf{M}^4)$ , and*

$$\begin{cases} \square_{\mathbf{g}} \mathcal{S}_{\alpha_1 \dots \alpha_k} = \mathcal{S}_{\beta_1 \dots \beta_k} \mathcal{A}^{\beta_1 \dots \beta_k}_{\alpha_1 \dots \alpha_k} + \mathbf{D}_{\beta_{k+1}} \mathcal{S}_{\beta_1 \dots \beta_k} \mathcal{B}^{\beta_1 \dots \beta_{k+1}}_{\alpha_1 \dots \alpha_k}; \\ \mathcal{L}_{\xi} \mathcal{S}_{\alpha_1 \dots \alpha_k} = \mathcal{S}_{\beta_1 \dots \beta_k} \mathcal{C}^{\beta_1 \dots \beta_k}_{\alpha_1 \dots \alpha_k}, \end{cases} \quad (1.6)$$

*in  $\mathbf{E}^4$ . Assume in addition that  $\mathcal{S} \equiv 0$  on  $\mathcal{H}$ . Then,  $\mathcal{S} \equiv 0$  on  $\mathbf{E}^4 \cup \mathcal{H}$ .*

## 2. UNIQUE CONTINUATION AND CONDITIONAL CARLEMAN INEQUALITIES

**2.1. General considerations.** Our proof of Theorem 1.2 is based on a global unique continuation strategy. We say that a linear differential operator  $L$ , in a domain  $\Omega \subset \mathbb{R}^d$ , satisfies the unique continuation property with respect to a smooth, oriented, hypersurface  $\Sigma \subset \Omega$ , if any smooth solution of  $L\phi = 0$  which vanishes on one side of  $\Sigma$  must in fact vanish in a small neighborhood of  $\Sigma$ . Such a property depends, of course, on the interplay between the properties of the operator  $L$  and the hypersurface  $\Sigma$ . A classical result of Hörmander, see for example Chapter 28 in [13], provides sufficient conditions for a scalar linear equation which guarantee that the unique continuation property holds. In the particular case of the scalar wave equation,  $\square_{\mathbf{g}}\phi = 0$ , and a smooth surface  $\Sigma$  defined by the equation  $h = 0$ ,  $\nabla h \neq 0$ , Hörmander's pseudo-convexity condition takes the form,

$$\mathbf{D}^2 h(X, X) < 0 \quad \text{if} \quad \mathbf{g}(X, X) = \mathbf{g}(X, \mathbf{D}h) = 0 \quad (2.1)$$

at all points on the surface  $\Sigma$ , where we assume that  $\phi$  is known to vanish on the side of  $\Sigma$  corresponding to  $h < 0$ .

In our situation, we plan to apply the general philosophy of unique continuation to the covariant wave equation (see the first equation in (1.6)),

$$\square_{\mathbf{g}} \mathcal{S} = \mathcal{A} * \mathcal{S} + \mathcal{B} * \mathbf{D}\mathcal{S}. \quad (2.2)$$

We know that  $\mathcal{S}$  vanishes on the horizon  $\mathcal{H}$  and we would like to prove, by unique continuation, that  $\mathcal{S}$  vanishes in the entire domain of outer communication. In implementing such a strategy one encounters the following difficulties:

- (1) The horizon  $\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-$  is characteristic and not smooth in a neighborhood of the bifurcate sphere.
- (2) Even though one can show that an appropriate variant of Hörmander's pseudo-convexity condition holds true along the horizon, in a neighborhood of the bifurcate sphere, such a condition may fail to be true slightly away from the horizon, within the ergosphere region of the stationary space-time where  $\xi$  is space-like.

Problem (1) can be dealt with by exploiting the fact that the horizon is a bifurcate characteristic hypersurface, which, in particular, is sufficient to allow us to prove that higher order derivatives of  $\mathcal{S}$  vanish on the horizon. Problem (2) is more serious, in the case when  $a$  is not small compared to  $m$ , because of the existence of null geodesics trapped within the ergoregion  $m + \sqrt{m^2 - a^2} \leq r \leq m + \sqrt{m^2 - a^2 \cos^2 \theta}$ . Indeed surfaces of the

form  $r\Delta = m(r^2 - a^2)^{1/2}$ , which intersect the ergoregion for  $a$  sufficiently close to  $m$ , are known to contain such null geodesics, see [4]. One can show that the presence of trapped null geodesics invalidates Hörmander's pseudo-convexity condition. Thus, even in the case of the scalar wave equation  $\square_{\mathbf{g}}\phi = 0$  in such a Kerr metric, one cannot guarantee, by a classical unique continuation argument (in the absence of additional conditions) that  $\phi$  vanishes beyond a small neighborhood of the horizon.

In order to overcome this main difficulty we need to exploit the second identity in (1.6), namely

$$\mathcal{L}_{\mathbf{T}}\mathcal{S} = \mathcal{C} * \mathcal{S}. \quad (2.3)$$

Observe that (2.3) can, in principle, transform (2.2) into a much simpler elliptic problem, in any domain which lies strictly outside the ergoregion (where  $\xi$  is strictly time-like). Unfortunately this possible strategy is not available to us when  $a$  is not small compared to  $m$ , since, as we have remarked above, we cannot hope to extend the vanishing of  $\mathcal{S}$ , by a simple analogue of Hörmander's pseudo-convexity condition, beyond the first trapped null geodesics.

Our solution is to extend Hörmander's classical pseudo-convexity condition (2.1) to one which takes into account both equations (2.2) and (2.3) simultaneously. These considerations lead to the following qualitative,  $\xi$ -conditional, pseudo-convexity condition,

$$\begin{aligned} \xi(h) &= 0; \\ \mathbf{D}^2h(X, X) &< 0 \quad \text{if} \quad \mathbf{g}(X, X) = \mathbf{g}(X, \mathbf{D}h) = \mathbf{g}(\xi, X) = 0. \end{aligned} \quad (2.4)$$

We will show that this condition can be verified in all Kerr spaces  $a \in [0, m)$ , for the simple function  $h = r$ , where  $r$  is one of the Boyer–Lindquist coordinates. Thus (2.4) is a good substitute for the more general condition (2.1). The fact that the two geometric identities (2.2) and (2.3) cooperate exactly in the right way, via (2.4), thus allowing us to compensate for both the failure of condition (2.1) as well as the failure of the vector field  $\xi$  to be time-like in the ergoregion, seems to us to be a very remarkable property of the Kerr spaces. In the next subsection we give a quantitative version of the condition and state a Carleman estimate of sufficient generality to cover all our needs.

**2.2. A conditional Carleman inequality of sufficient generality.** Unique continuation properties are often proved using Carleman inequalities. In this subsection we state a sufficiently general Carleman inequality, Proposition 2.3, under a quantitative conditional pseudo-convexity assumption. This general Carleman inequality is used to show first that  $\mathcal{S}$  vanishes in a small neighborhood of the bifurcate sphere  $S_0$  in  $\overline{\mathbf{E}^4}$ , using only the first identity in (1.6), and then to prove that  $\mathcal{S}$  vanishes in the entire exterior domain using both identities in (1.6). The two applications are genuinely different, since, in particular, the horizon is a bifurcate surface which is not smooth and the weights needed in this case have to be “singular” in an appropriate sense. In order to be able to cover both applications and prove unique continuation in a quantitative sense, we work with a more technical notion of conditional pseudo-convexity than (2.4), see Definition 2.1 below.

Let  $B_r = \{x \in \mathbb{R}^4 : |x| < r\}$  denote the standard open ball in  $\mathbb{R}^4$ . Assume that  $(M, \mathbf{g})$  is a smooth Lorentzian manifold of dimension 4,  $x_0 \in M$ , and  $\Phi^{x_0} : B_1 \rightarrow B_1(x_0)$  is a coordinate chart. For simplicity of notation, let  $B_r(x_0) = \Phi^{x_0}(B_r)$ ,  $r \in (0, 1]$ . For any smooth function  $\phi : B \rightarrow \mathbb{C}$ , where  $B \subseteq B_1(x_0)$  is an open set, and  $j = 0, 1, \dots$  let

$$|D^j \phi(x)| = \sum_{\alpha_1, \dots, \alpha_j=1}^4 |\partial_{\alpha_1} \dots \partial_{\alpha_j} \phi(x)|. \quad (2.5)$$

Let  $\mathbf{g}_{\alpha\beta} = \mathbf{g}(\partial_\alpha, \partial_\beta)$  and assume that  $V = V^\alpha \partial_\alpha$  is a vector-field on  $B_1(x_0)$ . We assume that

$$\sup_{x \in B_1(x_0)} \sum_{j=0}^6 \sum_{\alpha, \beta=1}^4 |D^j \mathbf{g}_{\alpha\beta}| + |D^j \mathbf{g}^{\alpha\beta}| + |D^j V^\beta| \leq A_0. \quad (2.6)$$

In our applications  $V = 0$  or  $V = \xi$ .

**Definition 2.1.** A family of weights  $h_\epsilon : B_{\epsilon^{10}}(x_0) \rightarrow \mathbb{R}_+$ ,  $\epsilon \in (0, \epsilon_1)$ ,  $\epsilon_1 \leq A_0^{-1}$ , will be called *V*-conditional pseudo-convex if for any  $\epsilon \in (0, \epsilon_1)$

$$h_\epsilon(x_0) = \epsilon, \quad \sup_{x \in B_{\epsilon^{10}}(x_0)} \sum_{j=1}^4 \epsilon^j |D^j h_\epsilon(x)| \leq \epsilon/\epsilon_1, \quad |V(h_\epsilon)(x_0)| \leq \epsilon^{10}, \quad (2.7)$$

$$\mathbf{D}^\alpha h_\epsilon(x_0) \mathbf{D}^\beta h_\epsilon(x_0) (\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) \geq \epsilon_1^2, \quad (2.8)$$

and there is  $\mu \in [-\epsilon_1^{-1}, \epsilon_1^{-1}]$  such that for all vectors  $X = X^\alpha \partial_\alpha \in \mathbb{T}_{x_0}(M)$

$$\begin{aligned} & \epsilon_1^2 [(X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2] \\ & \leq X^\alpha X^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2} (|X^\alpha V_\alpha(x_0)|^2 + |X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2). \end{aligned} \quad (2.9)$$

A function  $e_\epsilon : B_{\epsilon^{10}}(x_0) \rightarrow \mathbb{R}$  will be called a negligible perturbation if

$$\sup_{x \in B_{\epsilon^{10}}(x_0)} |D^j e_\epsilon(x)| \leq \epsilon^{10} \quad \text{for } j = 0, \dots, 4. \quad (2.10)$$

**Remark 2.2.** One can see that the technical conditions (2.7), (2.8), and (2.9) are related to the qualitative condition (2.4), at least when  $h_\epsilon = h + \epsilon$  for some smooth function  $h$ . The assumption  $|V(h_\epsilon)(x_0)| \leq \epsilon^{10}$  is a quantitative version of  $V(h) = 0$ . The assumption (2.9) is a quantitative version of the inequality in the second line of (2.4), in view of the large factor  $\epsilon^{-2}$  on the terms  $|X^\alpha V_\alpha(x_0)|^2$  and  $|X^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2$ , and the freedom to choose  $\mu$  in a large range. The assumption (2.8) is a quantitative version of the condition  $\nabla h \neq 0$  (assuming that (2.9) already holds).

It is important that the Carleman estimates we prove are stable under small perturbations of the weight, in order to be able to use them to prove unique continuation. We quantify this stability in (2.10).

We observe that if  $\{h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$  is a *V*-conditional pseudo-convex family, and  $e_\epsilon$  is a negligible perturbation for any  $\epsilon \in (0, \epsilon_1]$ , then

$$h_\epsilon + e_\epsilon \in [\epsilon/2, 2\epsilon] \text{ in } B_{\epsilon^{10}}(x_0).$$

The pseudo-convexity conditions of Definition 2.1 are probably not as general as possible, but are suitable for our applications both in Proposition 3.2, with “singular” weights  $h_\epsilon$  and  $V = 0$ , and Proposition 3.3, with “smooth” weights  $h_\epsilon$  and  $V = \xi$ .

**Proposition 2.3.** *Assume  $\epsilon_1 \leq A_0^{-1}$ ,  $\{h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$  is a  $V$ -conditional pseudo-convex family, and  $e_\epsilon$  is a negligible perturbation for any  $\epsilon \in (0, \epsilon_1]$ , see Definition 2.1. Then there is  $\epsilon \in (0, \epsilon_1)$  sufficiently small and  $C_\epsilon$  sufficiently large such that for any  $\lambda \geq C_\epsilon$  and any  $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$*

$$\lambda \|e^{-\lambda f_\epsilon} \phi\|_{L^2} + \|e^{-\lambda f_\epsilon} |D^1 \phi|\|_{L^2} \leq C_\epsilon \lambda^{-1/2} \|e^{-\lambda f_\epsilon} \square_{\mathbf{g}} \phi\|_{L^2} + \epsilon^{-6} \|e^{-\lambda f_\epsilon} V(\phi)\|_{L^2}, \quad (2.11)$$

where  $f_\epsilon = \ln(h_\epsilon + e_\epsilon)$ .

As mentioned earlier, many Carleman estimates such as (2.11) are known, for the particular case  $V = 0$ . Optimal proofs are usually based on some version of the Fefferman-Phong inequality, as in [13, Chapter 28]. A self-contained, elementary proof of Proposition 2.3, using only simple integration by parts arguments is given in [14, Section 3] (see also Proposition 4.1 in section 4 for a similar proof in a simpler case). We also note that it is useful to be able to track quantitatively the size of the support of the functions for which Carleman estimates can be applied; in our notation, the value of  $\epsilon$  for which (2.11) holds depends only on the parameter  $\epsilon_1$ .

### 3. PROOF OF THEOREM 1.2

**3.1. The first Carleman inequality in Kerr spaces.** The horizon  $\mathcal{H}$  decomposes as

$$\mathcal{H} = \mathcal{H}^+ \cup \mathcal{H}^-,$$

where  $\mathcal{H}^+$  is the boundary of the black hole region and  $\mathcal{H}^-$  is the boundary of the white hole region. Let  $S_0 = \mathcal{H}^+ \cap \mathcal{H}^-$  denote the bifurcate sphere. In this section we prove a Carleman estimate for functions supported in a small neighborhood of the bifurcate sphere  $S_0$ .

We first construct two suitable defining functions for the surfaces  $\mathcal{H}^+$  and  $\mathcal{H}^-$ .

**Lemma 3.1.** *There is an open set  $\mathbf{O} \subseteq \mathbf{M}^4$ ,  $S_0 \subseteq \mathbf{O}$ , and smooth functions  $u, v : \mathbf{O} \rightarrow \mathbb{R}$  with the following properties:*

(a) *We have*

$$\begin{cases} \mathbf{E}^4 \cap \mathbf{O} = \{x \in \mathbf{O} : u(x) > 0 \text{ and } v(x) > 0\}; \\ \mathcal{H}^+ \cap \mathbf{O} = \{x \in \mathbf{O} : u(x) = 0\}; \\ \mathcal{H}^- \cap \mathbf{O} = \{x \in \mathbf{O} : v(x) = 0\}. \end{cases}$$

*In addition, the set  $\{x \in \mathbf{O} : u(x), v(x) \in [0, 1/2]\}$  is compact.*

(b) *With  $L_3 = g^{\alpha\beta} \partial_\alpha(u) \partial_\beta$ ,  $L_4 = g^{\alpha\beta} \partial_\alpha(v) \partial_\beta \in \mathbb{T}(\mathbf{O})$ ,*

$$\begin{cases} \mathbf{g}(L_3, L_3) = 0 \text{ on } \mathcal{H}^+ \cap \mathbf{O}; \\ \mathbf{g}(L_4, L_4) = 0 \text{ on } \mathcal{H}^- \cap \mathbf{O}; \\ \mathbf{g}(L_3, L_4) > 0 \text{ on } S_0. \end{cases} \quad (3.1)$$



(c) For any smooth function  $\phi : \mathbf{O} \rightarrow \mathbb{R}$  with the property that  $\phi \equiv 0$  on  $\mathcal{H}^+ \cap \mathbf{O}$ , there is a smooth function  $\phi' : \mathbf{O} \rightarrow \mathbb{R}$  such that

$$\phi = \phi' \cdot u \text{ on } \mathbf{O} \cap \mathbf{E}^4.$$

Also, for any smooth function  $\phi : \mathbf{O} \rightarrow \mathbb{R}$  with the property that  $\phi \equiv 0$  on  $\mathcal{H}^- \cap \mathbf{O}$ , there is a smooth function  $\phi' : \mathbf{O} \rightarrow \mathbb{R}$  such that

$$\phi = \phi' \cdot v \text{ on } \mathbf{O} \cap \mathbf{E}^4.$$

*Proof of Lemma 3.1.* A more precise construction of global optical functions  $u, v$  is given in [18]. In our problem we do not need this global construction; for simplicity we construct the functions  $u, v$  explicitly, using the Kruskal coordinates of the Kerr space-times. In standard Boyer-Lindquist coordinates  $(r, t, \theta, \phi) \in (r_+, \infty) \times \mathbb{R} \times (0, \pi) \times \mathbb{S}^1$ ,  $r_{\pm} = m \pm (m^2 - a^2)^{1/2}$  (see the appendix), the Kerr metric on the dense open subset  $\tilde{\mathbf{E}}^4$  of  $\mathbf{E}^4$  is

$$ds^2 = -\frac{\rho^2 \Delta}{\Sigma^2} (dt)^2 + \frac{\Sigma^2 (\sin \theta)^2}{\rho^2} (d\phi - \omega dt)^2 + \frac{\rho^2}{\Delta} (dr)^2 + \rho^2 (d\theta)^2, \quad (3.2)$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2 (\cos \theta)^2; \\ \Sigma^2 = (r^2 + a^2) \rho^2 + 2mra^2 (\sin \theta)^2 = (r^2 + a^2)^2 - a^2 (\sin \theta)^2 \Delta; \\ \omega = \frac{2amr}{\Sigma^2}. \end{cases} \quad (3.3)$$

We define the function  $r_* : (r_+, \infty) \rightarrow \mathbb{R}$ ,

$$r_* = \int \frac{r^2 + a^2}{r^2 + a^2 - 2mr} dr = r + \frac{2mr_+}{r_+ - r_-} \ln(r - r_+) - \frac{2mr_-}{r_+ - r_-} \ln(r - r_-). \quad (3.4)$$

With  $c_0 = \frac{2mr_+}{r_+ - r_-}$ , we make the changes of variables

$$r_* = c_0 (\ln u + \ln v) \text{ and } t = c_0 (\ln u - \ln v), \quad (3.5)$$

where  $u, v \in (0, \infty)^2$ , so

$$\begin{cases} dr_* = c_0 (u^{-1} du + v^{-1} dv); \\ dt = c_0 (u^{-1} du - v^{-1} dv). \end{cases} \quad (3.6)$$

We observe also that  $\omega(r_+, \theta) = a/(2mr_+)$ . We make the change of variables

$$\phi = \phi_* + \frac{a}{2mr_+} t = \phi_* + \frac{ac_0}{2mr_+} (\ln u - \ln v), \quad (3.7)$$

with

$$d\phi = d\phi_* + \frac{ac_0}{2mr_+} (u^{-1} du - v^{-1} dv). \quad (3.8)$$

In the new coordinates  $(u, v, \theta, \phi_*) \in (0, \infty) \times (0, \infty) \times (0, \pi) \times \mathbb{S}^1$  the Kerr metric (3.2) becomes

$$ds^2 = -\frac{c_0^2 \rho^2 \Delta^2 a^2 (\sin \theta)^2}{u^2 v^2 \Sigma^2 (r^2 + a^2)^2} [v^2 (du)^2 + u^2 (dv)^2] + \frac{2c_0^2 \rho^2 \Delta}{uv} \left( \frac{1}{\Sigma^2} + \frac{1}{(r^2 + a^2)^2} \right) dudv \\ + \frac{\Sigma^2 (\sin \theta)^2}{\rho^2} \left[ d\phi_* - \frac{c_0 \tilde{\omega}}{uv} (vdu - u dv) \right]^2 + \rho^2 (d\theta)^2. \quad (3.9)$$

where  $\tilde{\omega} = \omega - a/(2mr_+)$ .

We restrict to the region

$$\tilde{\mathbf{O}} = \{(u, v, \theta, \phi_*) \in (-c_1, 1)^2 \times (0, \pi) \times \mathbb{S}^1\},$$

for some constant  $c_1 > 0$  sufficiently small. We examine the coefficients that appear in the Kerr metric (3.9). Since  $e^{r_*/c_0} = uv$  and  $r_- < r_+$  (since  $a \in [0, m)$ ), it follows from (3.4) that  $r$  is a smooth function of  $uv$  in  $\tilde{\mathbf{O}}$ . Moreover  $\Delta/(uv) = (r - r_-)(r - r_+)/(uv)$  and  $\tilde{\omega}/(uv)$  are smooth function of  $uv$  in  $\tilde{\mathbf{O}}$ . Thus the Kerr metric (3.9) is smooth in  $\tilde{\mathbf{O}}$ , and we identify  $\tilde{\mathbf{O}}$  with the corresponding open subset of the Kerr space. We let  $\mathbf{O}$  be any open neighborhood of  $S_0$  contained in the closure of  $\tilde{\mathbf{O}}$  in  $\mathbf{M}^4$  (by adding in the points corresponding to  $\theta \in \{0, \pi\}$ ). It is easy to see that the coordinate functions  $u, v : \mathbf{O} \rightarrow (-c_1, 1)$  verify the conclusions of the lemma.  $\square$

Assume now that  $x_0 \in S_0$ ,  $B_r = \{x \in \mathbb{R}^4 : |x| < r\}$ , and  $\Phi^{x_0} : B_1 \rightarrow \mathbf{O}$ ,  $\Phi^{x_0}(0) = x_0$ , is a smooth coordinate chart around  $x_0$ . In view of (3.1)

$$\delta_0 = \inf_{S_0} \mathbf{g}(L_3, L_4) > 0. \quad (3.10)$$

It follows from (3.1) that there is  $\epsilon_0 \in (0, 1/2]$  such that

$$\mathbf{g}(L_3, L_4) > \delta_0/2 \text{ and } |\mathbf{g}(L_3, L_3)| + |\mathbf{g}(L_4, L_4)| < \delta_0/100 \text{ on } B_{\epsilon_0}(x_0), \quad (3.11)$$

where  $B_r(x_0) = \Phi^{x_0}(B_r)$ . Thus we can fix smooth vector fields  $L_1, L_2 \in \mathbb{T}(B_{\epsilon_0}(x_0))$  such that

$$\mathbf{g}(L_1, L_1) = \mathbf{g}(L_2, L_2) = 1; \\ \mathbf{g}(L_1, L_2) = \mathbf{g}(L_1, L_3) = \mathbf{g}(L_2, L_3) = \mathbf{g}(L_1, L_4) = \mathbf{g}(L_2, L_4) = 0. \quad (3.12)$$

We define also the smooth function  $N^{x_0} : B_1(x_0) \rightarrow [0, \infty)$

$$N^{x_0}(x) = |(\Phi^{x_0})^{-1}(x)|^2.$$

The main result in this section is the following Carleman estimate:

**Proposition 3.2.** *There is  $\epsilon \in (0, \epsilon_0)$  sufficiently small and  $C_\epsilon$  sufficiently large such that for any  $\lambda \geq C_\epsilon$  and any  $\phi \in C_0^\infty(B_{\epsilon_{10}}(x_0))$*

$$\lambda \|e^{-\lambda f_\epsilon} \phi\|_{L^2} + \|e^{-\lambda f_\epsilon} |D^1 \phi|\|_{L^2} \leq C_\epsilon \lambda^{-1/2} \|e^{-\lambda f_\epsilon} \square_{\mathbf{g}} \phi\|_{L^2}, \quad (3.13)$$

where

$$f_\epsilon = \ln[\epsilon^{-1}(u + \epsilon)(v + \epsilon) + \epsilon^{12} N^{x_0}]. \quad (3.14)$$

*Proof of Proposition 3.2.* We apply Proposition 2.3 with  $V = 0$ . It is clear that  $\epsilon^{12}N^{x_0}$  is a negligible perturbation, in the sense of (2.10), for  $\epsilon$  sufficiently small. It remains to prove that there is  $\epsilon_1 > 0$  such that the family of weights  $\{h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$ ,

$$h_\epsilon = \epsilon^{-1}(u + \epsilon)(v + \epsilon) \quad (3.15)$$

satisfies conditions (2.7), (2.8) and (2.9).

Let  $\tilde{C}$  denote constants  $\geq 1$  that may depend only on the predefined geometric quantities  $\epsilon_0$ ,  $\delta_0$ , and a uniform bound in  $B_{\epsilon_0}(x_0)$  of  $|D^j \mathbf{g}_{\alpha\beta}|$ ,  $|D^j \mathbf{g}^{\alpha\beta}|$ ,  $|D^j u|$ ,  $|D^j v|$ ,  $j = 0, \dots, 6$ . Since  $u(x_0) = v(x_0) = 0$ , the definition (3.15) shows easily that condition (2.7) is satisfied, provided that  $\epsilon_1 \leq \tilde{C}^{-1}$ .

Relative to the frame  $L_1, L_2, L_3, L_4$  the metric  $\mathbf{g}$  takes the form,

$$\begin{cases} \mathbf{g}_{ab} = \delta_{ab}, & \mathbf{g}_{a3} = \mathbf{g}_{a4} = 0, & a, b = 1, 2 \\ \mathbf{g}_{33} = g_3, & \mathbf{g}_{44} = g_4, & \mathbf{g}_{34} = \Omega, \end{cases} \quad (3.16)$$

in  $B_{\epsilon_0}(x_0)$ , where  $g_3 = \mathbf{g}(L_3, L_3)$ ,  $g_4 = \mathbf{g}(L_4, L_4)$ ,  $\Omega = \mathbf{g}(L_3, L_4)$ . Also, for the inverse metric,

$$\begin{cases} \mathbf{g}^{ab} = \delta^{ab}, & \mathbf{g}^{a3} = \mathbf{g}^{a4} = 0, & a, b = 1, 2 \\ \mathbf{g}^{33} = g'_3, & \mathbf{g}^{44} = g'_4, & \mathbf{g}^{34} = \Omega', \end{cases} \quad (3.17)$$

where  $g'_3 = -g_4/(\Omega^2 - g_3g_4)$ ,  $g'_4 = -g_3/(\Omega^2 - g_3g_4)$ ,  $\Omega' = \Omega/(\Omega^2 - g_3g_4)$ . Recall that  $\Omega \geq \delta_0/2$  in  $B_{\epsilon_0}(x_0)$ , see (3.11),  $g_3 = 0$  on  $\mathcal{H}^+ \cap B_{\epsilon_0}(x_0)$ ,  $g_4 = 0$  on  $\mathcal{H}^- \cap B_{\epsilon_0}(x_0)$ , see (3.1). Thus, using Lemma 3.1 (c),

$$|g_3| \leq \tilde{C}u \quad \text{and} \quad |g_4| \leq \tilde{C}v \quad \text{in } B_{\epsilon_0}(x_0). \quad (3.18)$$

We denote by  $O(1)$  any quantity with absolute value bounded by a constant  $\tilde{C}$  as before. In view of the definitions of  $u, v, L_1, L_2, L_3, L_4$  we have,

$$L_1(u) = L_2(u) = L_1(v) = L_2(v) = 0, \quad L_3(u) = g_3, \quad L_4(v) = g_4, \quad L_4(u) = L_3(v) = \Omega. \quad (3.19)$$

Thus

$$\begin{aligned} L_4(h_\epsilon) &= \epsilon^{-1}(v + \epsilon)\Omega + \epsilon^{-1}(u + \epsilon)g_4, & L_3(h_\epsilon) &= \epsilon^{-1}(u + \epsilon)\Omega + \epsilon^{-1}(v + \epsilon)g_3, \\ L_1(h_\epsilon) &= L_2(h_\epsilon) = 0, \end{aligned} \quad (3.20)$$

and, using (3.18), (3.19), and (3.20), in  $B_{\epsilon_{10}}(x_0)$ ,

$$\begin{cases} (\mathbf{D}^2 h_\epsilon)_{34} = (\mathbf{D}^2 h_\epsilon)_{43} = \epsilon^{-1}\Omega^2 + O(1), \\ (\mathbf{D}^2 h_\epsilon)_{33} = O(1), & (\mathbf{D}^2 h_\epsilon)_{44} = O(1), & (\mathbf{D}^2 h_\epsilon)_{ab} = O(1), & a, b = 1, 2, \\ (\mathbf{D}^2 h_\epsilon)_{3a} = O(1), & (\mathbf{D}^2 h_\epsilon)_{4a} = O(1), & a = 1, 2. \end{cases} \quad (3.21)$$

Using (3.17), (3.20), (3.21), and  $g_3(x_0) = g_4(x_0) = 0$  we compute

$$\mathbf{D}^\alpha h_\epsilon(x_0) \mathbf{D}^\beta h_\epsilon(x_0) (\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) = 2\Omega^2 + \epsilon O(1) \geq \delta_0^2$$

if  $\epsilon_1$  is sufficiently small. Thus condition (2.8) is satisfied provided  $\epsilon_1 \leq \tilde{C}^{-1}$ .

Assume now  $Y = Y^\alpha L_\alpha$  is a vector in  $\mathbb{T}_{x_0}(\mathbf{M}^4)$ . We fix  $\mu = \epsilon_1^{-1/2}$  and compute, using (3.20), (3.21), and  $g_3(x_0) = g_4(x_0) = 1$ ,

$$\begin{aligned} & Y^\alpha Y^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2} |Y^\alpha \mathbf{D}_\alpha h_\epsilon|^2 \\ &= \mu((Y^1)^2 + (Y^2)^2 + 2\Omega Y^3 Y^4) - 2\epsilon^{-1} \Omega^2 Y^3 Y^4 + \epsilon^{-2} \Omega^2 (Y^3 + Y^4)^2 + O(1) \sum_{\alpha=1}^4 (Y^\alpha)^2 \\ &\geq (\mu/2)[(Y^1)^2 + (Y^2)^2] + \Omega^2 (\epsilon^{-1}/2)[(Y^3)^2 + (Y^4)^2] \\ &\geq (Y^1)^2 + (Y^2)^2 + (Y^3)^2 + (Y^4)^2 \end{aligned}$$

if  $\epsilon_1$  is sufficiently small. We notice now that we can write  $Y = X^\alpha \partial_\alpha$  in the coordinate frame  $\partial_1, \partial_2, \partial_3, \partial_4$ , and  $|X^\alpha| \leq \tilde{C}(|Y^1| + |Y^2| + |Y^3| + |Y^4|)$  for  $\alpha = 1, 2, 3, 4$ . Thus condition (2.9) is satisfied provided  $\epsilon_1 \leq \tilde{C}^{-1}$ , which completes the proof of the lemma.  $\square$

**3.2. The second Carleman inequality in Kerr spaces.** In this section we prove a Carleman estimate for functions supported in small open sets in  $\mathbf{E}^4$ . Assume that  $x_0 \in \mathbf{E}^4$  and  $\Phi^{x_0} : B_1 \rightarrow \mathbf{E}^4$ ,  $\Phi^{x_0}(0) = x_0$ , is a smooth coordinate chart around  $x_0$ . We define the smooth function  $N^{x_0} : B_1(x_0) \rightarrow [0, \infty)$ ,  $N^{x_0}(x) = |(\Phi^{x_0})^{-1}(x)|^2$  as before.

We use the notation in the appendix. The coordinate function  $r : \tilde{\mathbf{E}}^4 \rightarrow (r_+, \infty)$  extends to a smooth function  $r : \mathbf{E}^4 \rightarrow (r_+, \infty)$ . The main result in this subsection is the following Carleman estimate:

**Proposition 3.3.** *There is  $\epsilon \in (0, 1/2]$  sufficiently small and  $\tilde{C}_\epsilon$  sufficiently large such that for any  $\lambda \geq \tilde{C}_\epsilon$  and any  $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$*

$$\lambda \|e^{-\lambda \tilde{f}_\epsilon} \phi\|_{L^2} + \|e^{-\lambda \tilde{f}_\epsilon} |D^1 \phi|\|_{L^2} \leq \tilde{C}_\epsilon \lambda^{-1/2} \|e^{-\lambda \tilde{f}_\epsilon} \square_{\mathbf{g}} \phi\|_{L^2} + \epsilon^{-6} \|e^{-\lambda \tilde{f}_\epsilon} \xi(\phi)\|_{L^2}, \quad (3.22)$$

where, with  $r_0 = r(x_0)$ ,

$$\tilde{f}_\epsilon = \ln[r - r_0 + \epsilon + \epsilon^{12} N^{x_0}]. \quad (3.23)$$

*Proof of Proposition 3.3.* As in the proof of Proposition 3.2, we will use the notation  $\tilde{C}$  to denote various constants in  $[1, \infty)$  that may depend only on the chart  $\Phi$  and the position of  $x_0$  in  $\mathbf{E}^4$  (i.e. on  $(r(x_0) - r_+)^{-1} + (r(x_0) - r_+)$ ), and  $O(1)$  to denote quantities bounded in absolute value by a constant  $\tilde{C}$ . It is important to keep in mind that  $r(x_0) > r_+$ , i.e.  $x_0 \in \mathbf{E}^4$ . We apply Proposition 2.3 with  $V = \xi$ . It suffices to prove that there is  $\epsilon_1 > 0$  such that the family of weights  $h_\epsilon\}_{\epsilon \in (0, \epsilon_1)}$ ,

$$h_\epsilon = r - r_0 + \epsilon \quad (3.24)$$

satisfies conditions (2.7), (2.8), and (2.9).

Condition (2.7) is clear if  $\epsilon_1$  is sufficiently small, since  $\xi(h_\epsilon) = 0$ . To prove conditions (2.8) and (2.9), with the notation in section A, we work in the orthonormal frame  $e_0, e_1, e_2, e_3$  defined in (A.7). We have

$$\mathbf{D}_0(h_\epsilon) = \mathbf{D}_1(h_\epsilon) = \mathbf{D}_3(h_\epsilon) = 0, \quad \mathbf{D}_2(h_\epsilon) = (\Delta/\rho^2)^{1/2}. \quad (3.25)$$

Using the table (A.16), we have

$$\begin{aligned}
-\mathbf{D}_0\mathbf{D}_0h_\epsilon &= \frac{\Delta}{\rho^2} \left( \frac{r}{\rho^2} + \frac{r-m}{\Delta} - \frac{Y}{\Sigma^2} \right) \\
-\mathbf{D}_0\mathbf{D}_1h_\epsilon &= -\frac{\Delta}{\rho^2} \cdot \frac{ma \sin \theta}{\rho^2 \sqrt{\Delta} \Sigma^2} (2rY - \Sigma^2) \\
-\mathbf{D}_1\mathbf{D}_1h_\epsilon &= -\frac{\Delta}{\rho^2} \left( \frac{Y}{\Sigma^2} - \frac{r}{\rho^2} \right) \\
-\mathbf{D}_2\mathbf{D}_2h_\epsilon &= \frac{\Delta}{\rho^2} \left( \frac{r}{\rho^2} - \frac{r-m}{\Delta} \right) \\
-\mathbf{D}_2\mathbf{D}_3h_\epsilon &= -\frac{\sqrt{\Delta} a^2 \sin \theta \cos \theta}{\rho^4} \\
-\mathbf{D}_3\mathbf{D}_3h_\epsilon &= -\frac{\Delta r}{\rho^4} \\
\mathbf{D}_0\mathbf{D}_2h_\epsilon &= \mathbf{D}_0\mathbf{D}_3h_\epsilon = \mathbf{D}_1\mathbf{D}_2h_\epsilon = \mathbf{D}_1\mathbf{D}_3h_\epsilon = 0.
\end{aligned} \tag{3.26}$$

It follows that

$$\mathbf{D}^\alpha h_\epsilon(x_0) \mathbf{D}^\beta h_\epsilon(x_0) (\mathbf{D}_\alpha h_\epsilon \mathbf{D}_\beta h_\epsilon - \epsilon \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) = \Delta^2 / \rho^2 + \epsilon O(1)$$

which verifies condition (2.8) if  $\epsilon_1$  is sufficiently small.

To verify condition (2.9) we fix

$$\mu = \frac{3\Delta r}{2\rho^4} \tag{3.27}$$

and use the formula (compare with (A.9) and (A.4))

$$\xi = \frac{\rho \sqrt{\Delta}}{\Sigma} e_0 - \frac{2amr \sin \theta}{\rho \Sigma} e_1. \tag{3.28}$$

Assume  $X = Y^0 e_0 + Y^1 e_1 + Y^2 e_2 + Y^3 e_3$  is a vector expressed in the frame  $e_\alpha$ . We compute

$$\begin{aligned}
& Y^\alpha Y^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2} (|Y^\alpha \xi_\alpha(x_0)|^2 + |Y^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2) \\
&= (Y^0)^2 (-\mu - \mathbf{D}_0 \mathbf{D}_0 h_\epsilon) + (Y^1)^2 (\mu - \mathbf{D}_1 \mathbf{D}_1 h_\epsilon) + 2Y^0 Y^1 (-\mathbf{D}_0 \mathbf{D}_1 h_\epsilon) \\
&+ (Y^2)^2 (\mu - \mathbf{D}_2 \mathbf{D}_2 h_\epsilon) + (Y^3)^2 (\mu - \mathbf{D}_3 \mathbf{D}_3 h_\epsilon) + 2Y^2 Y^3 (-\mathbf{D}_2 \mathbf{D}_3 h_\epsilon) \\
&+ \epsilon^{-2} \frac{(\rho^2 \sqrt{\Delta} Y^0 + 2amr(\sin \theta) Y^1)^2}{\rho^2 \Sigma^2} + \epsilon^{-2} \frac{\Delta (Y^2)^2}{\rho^4}.
\end{aligned} \tag{3.29}$$

Let  $Z = \rho^2 \sqrt{\Delta} Y^0 + 2amr(\sin \theta) Y^1$ , thus

$$Y^0 = \frac{Z - 2amr(\sin \theta) Y^1}{\rho^2 \sqrt{\Delta}} = \alpha Y^1 + \beta Z.$$

Using also  $\mu - \mathbf{D}_3 \mathbf{D}_3 h_\epsilon = (\Delta r)/(2\rho^4)$ , the right-hand side of (3.29) becomes

$$\begin{aligned} & (Y^2)^2(\epsilon^{-2}\Delta\rho^{-4} + \mu - \mathbf{D}_2 \mathbf{D}_2 h_\epsilon) + (Y^3)^2(\Delta r \rho^{-4})/2 - 2Y^2 Y^3 \cdot \mathbf{D}_2 \mathbf{D}_3 h_\epsilon \\ & + Z^2[\epsilon^{-2}\rho^{-2}\Sigma^{-2} + \beta^2(-\mu - \mathbf{D}_0 \mathbf{D}_0 h_\epsilon)] \\ & + (Y^1)^2[\alpha^2(-\mu - \mathbf{D}_0 \mathbf{D}_0 h_\epsilon) - 2\alpha \mathbf{D}_0 \mathbf{D}_1 h_\epsilon + \mu - \mathbf{D}_1 \mathbf{D}_1 h_\epsilon] \\ & + 2Y^1 Z[\alpha\beta(-\mu - \mathbf{D}_0 \mathbf{D}_0 h_\epsilon) - \beta \mathbf{D}_0 \mathbf{D}_1 h_\epsilon]. \end{aligned} \tag{3.30}$$

It is clear that the first line of the expression above is bounded from below by

$$\tilde{C}^{-1}(\epsilon^{-2}(Y^2)^2 + (Y^3)^2)$$

if  $\epsilon$  is sufficiently small, since  $\Delta \geq \tilde{C}^{-1}$ . The main term we need to bound from below is the coefficient of  $(Y^1)^2$  in (3.30). We use the table (3.26) and the definitions of  $\alpha$  and  $\mu$ ; after several simplifications this term is equal to

$$\frac{5\Delta r}{2\rho^4} - \frac{\Delta Y}{\rho^2 \Sigma^2} + \frac{4a^2 m^2 r (\sin \theta)^2}{\rho^6} \left( -\frac{r^2}{2\rho^2} + \frac{rY}{\Sigma^2} + \frac{mr - a^2}{\Delta} \right).$$

In view of (A.14) and (A.15) this is bounded from below by  $(\Delta r)/(2\rho^4)$ . Thus the sum of the last three lines of (3.30) is bounded from below by

$$\tilde{C}^{-1}(\epsilon^{-2}Z^2 + (Y^1)^2)$$

if  $\epsilon$  is sufficiently small. It follows that

$$\begin{aligned} & Y^\alpha Y^\beta (\mu \mathbf{g}_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta h_\epsilon)(x_0) + \epsilon^{-2}(|Y^\alpha \xi_\alpha(x_0)|^2 + |Y^\alpha \mathbf{D}_\alpha h_\epsilon(x_0)|^2) \\ & \geq \tilde{C}^{-1}[(Y^0)^2 + (Y^1)^2 + \epsilon^{-2}(Y^2)^2 + (Y^3)^2] \end{aligned}$$

if  $\epsilon$  is sufficiently small. The condition (2.9) is verified, which completes the proof of the proposition.  $\square$

**3.3. Vanishing of the tensor  $\mathcal{S}$ .** In this subsection we prove Theorem 1.2. Arguments showing how to use Carleman inequalities to prove uniqueness are standard. We provide all the details here for the sake of completeness. Some care is needed at the first step, in Lemma 3.4 below, since we do not assume that derivatives of the tensor  $\mathcal{S}$  vanish on the horizon.

We show first that the tensor  $\mathcal{S}$  vanishes in a neighborhood of the bifurcate sphere  $S_0$  in  $\mathbf{E}^4$ .

**Lemma 3.4.** *With the notation in Theorem 1.2, there is an open set  $\mathbf{O}' \subseteq \mathbf{M}^4$ ,  $S_0 \subseteq \mathbf{O}'$ , such that*

$$\mathcal{S} \equiv 0 \text{ in } \mathbf{O}' \cap \mathbf{E}^4.$$

*Proof of Lemma 3.4.* We use the functions  $u, v$  defined in Lemma 3.1 and the Carleman estimate in Proposition 3.2. Since  $S_0$  is compact, it suffices to prove that for every point  $x_0 \in S_0$  there is a neighborhood  $\mathbf{O}'_{x_0}$  of  $x_0$  such that  $\mathcal{S} \equiv 0$  in  $\mathbf{E}^4 \cap \mathbf{O}'_{x_0}$ . As in Proposition 3.2, assume  $\Phi^{x_0} : B_1 \rightarrow \mathbf{O}$ ,  $\Phi^{x_0}(0) = x_0$ , is a smooth coordinate chart around  $x_0$ . With

the notation in Proposition 3.2, there are constants  $\epsilon \in (0, \epsilon_0)$  and  $\tilde{C} \geq 1$  such that, for any  $\lambda \geq \tilde{C}$  and any  $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$

$$\lambda \|e^{-\lambda f_\epsilon} \phi\|_{L^2} + \|e^{-\lambda f_\epsilon} |D^1 \phi|\|_{L^2} \leq \tilde{C} \lambda^{-1/2} \|e^{-\lambda f_\epsilon} \square_{\mathbf{g}} \phi\|_{L^2}, \quad (3.31)$$

where

$$f_\epsilon = \ln[\epsilon^{-1}(u + \epsilon)(v + \epsilon) + \epsilon^{12} N^{x_0}]. \quad (3.32)$$

The constant  $\epsilon$  will remain fixed in this proof, and we assume implicitly it is sufficiently small as discussed in Proposition 3.2. We will show that

$$\mathcal{S} \equiv 0 \text{ in } B_{\epsilon^{40}}(x_0) \cap \mathbf{E}^4. \quad (3.33)$$

For  $(j_1, \dots, j_k) \in \{1, 2, 3, 4\}^k$  we define, using the coordinate chart  $\Phi$ ,

$$\phi_{(j_1 \dots j_k)} = \mathcal{S}(\partial_{j_1}, \dots, \partial_{j_k}). \quad (3.34)$$

If  $k = 0$  we define  $\phi = \mathcal{S}$  in  $B_1(x_0)$ . The functions  $\phi_{(j_1 \dots j_k)} : B_1(x_0) \rightarrow \mathbb{C}$  are smooth. Let  $\eta : \mathbb{R} \rightarrow [0, 1]$  denote a smooth function supported in  $[1/2, \infty)$  and equal to 1 in  $[3/4, \infty)$ . With  $u, v$  as in Proposition 3.1, for  $\delta \in (0, 1]$  we define

$$\begin{aligned} \phi_{(j_1 \dots j_k)}^{\delta, \epsilon} &= \phi_{(j_1 \dots j_k)} \cdot \mathbf{1}_{\mathbf{E}^4} \cdot \eta(uv/\delta) \cdot (1 - \eta(N^{x_0}/\epsilon^{20})) \\ &= \phi_{(j_1 \dots j_k)} \cdot \tilde{\eta}_{\delta, \epsilon}. \end{aligned} \quad (3.35)$$

Clearly,  $\phi_{(j_1 \dots j_k)}^{\delta, \epsilon} \in C_0^\infty(B_{\epsilon^{10}}(x_0))$ . We would like to apply the inequality (3.31) to the functions  $\phi_{(j_1 \dots j_k)}^{\delta, \epsilon}$ , and then let  $\delta \rightarrow 0$  and  $\lambda \rightarrow \infty$  (in this order).

Using the definition (3.35), we have

$$\square_{\mathbf{g}} \phi_{(j_1 \dots j_k)}^{\delta, \epsilon} = \tilde{\eta}_{\delta, \epsilon} \cdot \square_{\mathbf{g}} \phi_{(j_1 \dots j_k)} + 2\mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \cdot \mathbf{D}^\alpha \tilde{\eta}_{\delta, \epsilon} + \phi_{(j_1 \dots j_k)} \cdot \square_{\mathbf{g}} \tilde{\eta}_{\delta, \epsilon}.$$

Using the Carleman inequality (3.31), for any  $(j_1, \dots, j_k) \in \{1, 2, 3, 4\}^k$  we have

$$\begin{aligned} &\lambda \cdot \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta, \epsilon} \phi_{(j_1 \dots j_k)}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta, \epsilon} |D^1 \phi_{(j_1 \dots j_k)}|\|_{L^2} \\ &\leq \tilde{C} \lambda^{-1/2} \cdot \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta, \epsilon} \square_{\mathbf{g}} \phi_{(j_1 \dots j_k)}\|_{L^2} \\ &\quad + \tilde{C} \left[ \|e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \mathbf{D}^\alpha \tilde{\eta}_{\delta, \epsilon}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot \phi_{(j_1 \dots j_k)} (|\square_{\mathbf{g}} \tilde{\eta}_{\delta, \epsilon}| + |D^1 \tilde{\eta}_{\delta, \epsilon}|)\|_{L^2} \right], \end{aligned} \quad (3.36)$$

for any  $\lambda \geq \tilde{C}$ . We estimate now  $|\square_{\mathbf{g}} \phi_{(j_1 \dots j_k)}|$ . Using the first identity in (1.6) and (3.34), in  $B_{\epsilon^{10}}(x_0)$  we estimate pointwise

$$|\square_{\mathbf{g}} \phi_{(j_1 \dots j_k)}| \leq \tilde{C}_{\mathcal{A}, \mathcal{B}} \sum_{l_1, \dots, l_k} (|D^1 \phi_{(l_1 \dots l_k)}| + |\phi_{(l_1 \dots l_k)}|), \quad (3.37)$$

for some constant  $\tilde{C}_{\mathcal{A}, \mathcal{B}}$  that depends only on the tensors  $\mathcal{A}$  and  $\mathcal{B}$ . We add up the inequalities (3.36) over  $(j_1, \dots, j_k) \in \{1, 2, 3, 4\}^k$ . The key observation is that, in view of

(3.37), the first term in the right-hand side can be absorbed into the left-hand side for  $\lambda$  sufficiently large. Thus, for any  $\lambda \geq \tilde{C}_{\mathcal{A},\mathcal{B}}$  and  $\delta \in (0, 1]$

$$\begin{aligned} & \lambda \sum_{j_1, \dots, j_k} \|e^{-\lambda f_\epsilon} \cdot \tilde{\eta}_{\delta, \epsilon} \phi_{(j_1 \dots j_k)}\|_{L^2} \\ & \leq \tilde{C} \sum_{j_1, \dots, j_k} \left[ \|e^{-\lambda f_\epsilon} \cdot \mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \mathbf{D}^\alpha \tilde{\eta}_{\delta, \epsilon}\|_{L^2} + \|e^{-\lambda f_\epsilon} \cdot \phi_{(j_1 \dots j_k)} (|\square_{\mathbf{g}} \tilde{\eta}_{\delta, \epsilon}| + |D^1 \tilde{\eta}_{\delta, \epsilon}|)\|_{L^2} \right]. \end{aligned} \quad (3.38)$$

We would like to let  $\delta \rightarrow 0$  in (3.38). For this, we observe first that the functions  $\mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \mathbf{D}^\alpha \tilde{\eta}_{\delta, \epsilon}$  and  $(|\square_{\mathbf{g}} \tilde{\eta}_{\delta, \epsilon}| + |D^1 \tilde{\eta}_{\delta, \epsilon}|)$  vanish outside the set  $\mathbf{A}_\delta \cup \tilde{\mathbf{B}}_\epsilon$ , where

$$\begin{cases} \mathbf{A}_\delta = \{x \in B_{\epsilon^{10}}(x_0) \cap \mathbf{E}^4 : uv \in (\delta/2, \delta)\}; \\ \tilde{\mathbf{B}}_\epsilon = \{x \in B_{\epsilon^{10}}(x_0) \cap \mathbf{E}^4 : N^{x_0} \in (\epsilon^{20}/2, \epsilon^{20})\}. \end{cases}$$

In addition, since  $\phi_{(j_1 \dots j_k)} = 0$  on  $\mathcal{H}$  (using the hypothesis of Theorem 1.1), it follows from Proposition 3.1 (c) that there are smooth functions  $\phi'_{(j_1 \dots j_k)} : \mathbf{O} \rightarrow \mathbb{C}$  such that

$$\phi_{(j_1 \dots j_k)} (1 - \eta(N^{x_0})) = uv \cdot \phi'_{(j_1 \dots j_k)} \text{ in } \mathbf{O} \cap \mathbf{E}^4. \quad (3.39)$$

We show now that

$$|\square_{\mathbf{g}} \tilde{\eta}_{\delta, \epsilon}| + |D^1 \tilde{\eta}_{\delta, \epsilon}| \leq \tilde{C}(\mathbf{1}_{\tilde{\mathbf{B}}_\epsilon} + (1/\delta)\mathbf{1}_{\mathbf{A}_\delta}). \quad (3.40)$$

The inequality for  $|D^1 \tilde{\eta}_{\delta, \epsilon}|$  follows directly from the definition (3.35). Also, using again the definition,

$$|\mathbf{D}^\alpha \mathbf{D}_\alpha \tilde{\eta}_{\delta, \epsilon}| \leq |\mathbf{D}^\alpha \mathbf{D}_\alpha (\mathbf{1}_{\mathbf{E}^4} \cdot \eta(uv/\delta))| \cdot (1 - \eta(N^{x_0}/\epsilon^{20})) + \tilde{C}(\mathbf{1}_{\tilde{\mathbf{B}}_\epsilon} + (1/\delta)\mathbf{1}_{\mathbf{A}_\delta}).$$

Thus, for (3.40), it suffices to prove that

$$\mathbf{1}_{\mathbf{E}^4} \cdot |\mathbf{D}^\alpha \mathbf{D}_\alpha (\eta(uv/\delta))| \leq \tilde{C}/\delta \cdot \mathbf{1}_{\mathbf{A}_\delta}. \quad (3.41)$$

Since  $u, v, \eta$  are smooth functions, for (3.41) it suffices to prove that

$$\delta^{-2} |\mathbf{D}^\alpha (uv) \mathbf{D}_\alpha (uv)| \leq \tilde{C}/\delta \text{ in } \mathbf{A}_\delta. \quad (3.42)$$

Since  $uv \in [\delta/2, \delta]$  in  $\mathbf{A}_\delta$ , it suffices to prove that

$$u^2 |\mathbf{D}^\alpha v \mathbf{D}_\alpha v| + v^2 |\mathbf{D}^\alpha u \mathbf{D}_\alpha u| \leq \tilde{C}\delta \text{ in } \mathbf{A}_\delta.$$

For this we use the frame  $L_1, L_2, L_3, L_4$  as in the proof of Proposition 3.2. The bound follows from (3.19), (3.18), and (3.17).

We show now that

$$|\mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \mathbf{D}^\alpha \tilde{\eta}_{\delta, \epsilon}| \leq \tilde{C}_{\phi'} (\mathbf{1}_{\tilde{\mathbf{B}}_\epsilon} + \mathbf{1}_{\mathbf{A}_\delta}), \quad (3.43)$$

where the constant  $\tilde{C}_{\phi'}$  depends on the smooth functions  $\phi'_{(j_1 \dots j_k)}$  defined in (3.39). Using the formula (3.39) (which becomes  $\phi_{(j_1 \dots j_k)} = uv \cdot \phi'_{(j_1 \dots j_k)}$  in  $\mathbf{A}_\delta \cup \tilde{\mathbf{B}}_\epsilon$ ), this follows easily from (3.42).



It follows from (3.39), (3.40), and (3.43) that

$$|\mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \mathbf{D}^\alpha \tilde{\eta}_{\delta, \epsilon}| + |\phi_{j_1 \dots j_k}|(|\square_{\mathbf{g}} \tilde{\eta}_{\delta, \epsilon}| + |D^1 \tilde{\eta}_{\delta, \epsilon}|) \leq \tilde{C}_{\phi'}(\mathbf{1}_{\tilde{\mathbf{B}}_\epsilon} + \mathbf{1}_{\mathbf{A}_\delta}).$$

Since  $\lim_{\delta \rightarrow 0} \|\mathbf{1}_{\mathbf{A}_\delta}\|_{L^2} = 0$ , we can let  $\delta \rightarrow 0$  in (3.38) to conclude that

$$\lambda \sum_{j_1, \dots, j_k} \|e^{-\lambda f_\epsilon} \cdot \mathbf{1}_{B_{\epsilon^{10/2}}(x_0) \cap \mathbf{E}^4} \cdot \phi_{(j_1 \dots j_k)}\|_{L^2} \leq \tilde{C}_{\phi'} \sum_{j_1, \dots, j_k} \|e^{-\lambda f_\epsilon} \cdot \mathbf{1}_{\tilde{\mathbf{B}}_\epsilon}\|_{L^2} \quad (3.44)$$

for any  $\lambda \geq \tilde{C}_{\mathcal{A}, \mathcal{B}}$ . Finally, using the definition (3.32), we observe that

$$\inf_{B_{\epsilon^{40}}(x_0) \cap \mathbf{E}^4} e^{-\lambda f_\epsilon} \geq e^{-\lambda \ln(\epsilon + \epsilon^{32/2})} \geq \sup_{\tilde{\mathbf{B}}_\epsilon} e^{-\lambda f_\epsilon}.$$

It follows from (3.44) that

$$\lambda \sum_{j_1, \dots, j_k} \|\mathbf{1}_{B_{\epsilon^{40}}(x_0) \cap \mathbf{E}^4} \cdot \phi_{(j_1 \dots j_k)}\|_{L^2} \leq \tilde{C}_{\phi'} \sum_{j_1, \dots, j_k} \|\mathbf{1}_{\tilde{\mathbf{B}}_\epsilon}\|_{L^2}$$

for any  $\lambda \geq \tilde{C}_{\mathcal{A}, \mathcal{B}}$ . We let now  $\lambda \rightarrow \infty$ . The identity (3.33) follows.  $\square$

We show now that the tensor  $\mathcal{S}$  vanishes in an open neighborhood of the horizon  $\mathcal{H}$  in  $\mathbf{E}^4$ . For any  $R > r_+$  let

$$\mathbf{E}_R^4 = \{x \in \mathbf{E}^4 : r(x) \in (r_+, R)\},$$

where  $r : \mathbf{E}^4 \rightarrow (r_+, \infty)$  is the smooth function used in Proposition 3.3.

**Lemma 3.5.** *With the notation in Theorem 1.2, there is  $R > r_+$  such that*

$$\mathcal{S} \equiv 0 \text{ in } \mathbf{E}_R^4.$$

*Proof of Lemma 3.5.* It follows from Proposition 3.1 (a) and Lemma 3.4 that there is  $\epsilon_1 > 0$  such that

$$\mathcal{S} \equiv 0 \text{ in the set } \{x \in \mathbf{E}^4 \cap \mathbf{O} : u(x) < \epsilon_1 \text{ and } v(x) < \epsilon_1\}. \quad (3.45)$$

It suffices to prove that  $\mathcal{S} \equiv 0$  in  $\mathbf{E}_R^4 \cap \tilde{\mathbf{E}}^4$ , where  $\tilde{\mathbf{E}}^4$  is the dense open subset of  $\mathbf{E}^4$  defined in section A. In view of (3.45) and the definition of the functions  $u, v$  in the proof of Lemma 3.1, there is  $\epsilon_2 > 0$  such that

$$\mathcal{S} \equiv 0 \text{ in the set } \{x = (r, t, \theta, \phi) \in \tilde{\mathbf{E}}^4 : t = 0 \text{ and } r < r_+ + \epsilon_2\}. \quad (3.46)$$

We use the Boyer-Lindquist coordinate chart (see appendix A) to define

$$\tilde{\partial}_1 = \partial_r, \quad \tilde{\partial}_2 = \partial_t, \quad \tilde{\partial}_3 = \partial_\theta, \quad \tilde{\partial}_4 = \partial_\phi$$

and

$$\tilde{\phi}_{(j_1 \dots j_k)} = \mathcal{S}(\tilde{\partial}_{j_1}, \dots, \tilde{\partial}_{j_k})$$

The second identity in (1.6) gives, for any  $(j_1, \dots, j_k) \in \{1, 2, 3, 4\}^k$

$$\partial_t(\tilde{\phi}_{(j_1 \dots j_k)}) = \sum_{l_1, \dots, l_k} \tilde{\phi}_{(l_1 \dots l_k)} c^{l_1 \dots l_k}_{j_1 \dots j_k}. \quad (3.47)$$

In view of (3.46)

$$\tilde{\phi}_{(j_1 \dots j_k)}(r, 0, \theta, \phi) = 0 \text{ if } r < r_+ + \epsilon_2.$$

Since  $\mathcal{C}$  is a smooth tensor in  $\tilde{\mathbf{E}}^4$ , it follows that  $\tilde{\phi}_{(j_1 \dots j_k)}(r, t, \theta, \phi) = 0$  if  $r < r_+ + \epsilon_2$ , which completes the proof of the lemma.  $\square$

We prove now that  $\mathcal{S} \equiv 0$  in  $\mathbf{E}^4$ , which completes the proof of the theorem. In view of Lemma 3.5, it suffices to prove the following:

**Lemma 3.6.** *With the notation in Theorem 1.1, assume that*

$$\mathcal{S} \equiv 0 \text{ in } \mathbf{E}_{R_0}^4. \quad (3.48)$$

*for some  $R_0 > r_+$ . Then there is  $R_1 > R_0$  such that*

$$\mathcal{S} \equiv 0 \text{ in } \mathbf{E}_{R_1}^4.$$

*Proof of Lemma 3.6.* Assume that  $x_0 \in \mathbf{E}^4$  and  $r(x_0) = R_0$ . We show first that

$$\text{there is a neighborhood } \mathbf{O}'_{x_0} \text{ of } x_0 \text{ such that } \mathcal{S} \equiv 0 \text{ in } \mathbf{O}'_{x_0}. \quad (3.49)$$

This is similar to the proof of Lemma 3.4, using the Carleman estimate in Proposition 3.3 instead of the Carleman estimate in Proposition 3.2. Assume  $\Phi^{x_0} : B_1 \rightarrow \mathbf{E}^4$ ,  $\Phi^{x_0}(0) = x_0$ , is a smooth coordinate chart around  $x_0$ . With the notation in Proposition 3.3, there is  $\epsilon \in (0, 1/2]$  sufficiently small and  $\tilde{C}$  sufficiently large such that

$$\lambda \|e^{-\lambda \tilde{f}_\epsilon} \phi\|_{L^2} + \|e^{-\lambda \tilde{f}_\epsilon} |D^1 \phi|\|_{L^2} \leq \tilde{C} \lambda^{-1/2} \|e^{-\lambda \tilde{f}_\epsilon} \square_{\mathbf{g}} \phi\|_{L^2} + \epsilon^{-6} \|e^{-\lambda \tilde{f}_\epsilon} \xi(\phi)\|_{L^2}, \quad (3.50)$$

for any  $\lambda \geq \tilde{C}$  and any  $\phi \in C_0^\infty(B_{\epsilon^{10}}(x_0))$ , where

$$\tilde{f}_\epsilon = \ln[r - R_0 + \epsilon + \epsilon^{12} N^{x_0}]. \quad (3.51)$$

The constant  $\epsilon$  will remain fixed in this proof, and sufficiently small in the sense of Proposition 3.3. We will show that

$$\mathcal{S} \equiv 0 \text{ in } B_{\epsilon^{40}}(x_0). \quad (3.52)$$

For  $(j_1, \dots, j_k) \in \{1, 2, 3, 4\}^k$  we define, using the coordinate chart  $\Phi$ ,

$$\phi_{(j_1 \dots j_k)} = \mathcal{S}(\partial_{j_1}, \dots, \partial_{j_k}).$$

If  $k = 0$  we simply define  $\phi = \mathcal{S}$  in  $B_1(x_0)$ . The functions  $\phi_{(j_1 \dots j_k)} : B_1(x_0) \rightarrow \mathbb{C}$  are smooth. Let  $\eta : \mathbb{R} \rightarrow [0, 1]$  denote a smooth function supported in  $[1/2, \infty)$  and equal to 1 in  $[3/4, \infty)$ , as before. We define

$$\phi_{(j_1 \dots j_k)}^\epsilon = \phi_{(j_1 \dots j_k)} \cdot (1 - \eta(N^{x_0}/\epsilon^{20})) = \phi_{(j_1 \dots j_k)} \cdot \tilde{\eta}_\epsilon.$$

Clearly,  $\phi_{(j_1 \dots j_k)}^\epsilon \in C_0^\infty(B_{\epsilon^{10}}(x_0))$  and

$$\begin{cases} \square_{\mathbf{g}} \phi_{(j_1 \dots j_k)}^\epsilon = \tilde{\eta}_\epsilon \cdot \square_{\mathbf{g}} \phi_{(j_1 \dots j_k)} + 2\mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \cdot \mathbf{D}^\alpha \tilde{\eta}_\epsilon + \phi_{(j_1 \dots j_k)} \cdot \square_{\mathbf{g}} \tilde{\eta}_\epsilon \\ \xi(\phi_{(j_1 \dots j_k)}^\epsilon) = \tilde{\eta}_\epsilon \cdot \xi(\phi_{(j_1 \dots j_k)}) + \phi_{(j_1 \dots j_k)} \cdot \xi(\tilde{\eta}_\epsilon). \end{cases}$$

Using the Carleman inequality (3.50), for any  $(j_1, \dots, j_k) \in \{1, 2, 3, 4\}^k$  we have

$$\begin{aligned} & \lambda \cdot \|e^{-\lambda \tilde{f}_\epsilon} \cdot \tilde{\eta}_\epsilon \phi_{(j_1 \dots j_k)}\|_{L^2} + \|e^{-\lambda \tilde{f}_\epsilon} \cdot \tilde{\eta}_\epsilon |D^1 \phi_{(j_1 \dots j_k)}|\|_{L^2} \\ & \leq \tilde{C} \lambda^{-1/2} \cdot \|e^{-\lambda \tilde{f}_\epsilon} \cdot \tilde{\eta}_\epsilon \square_{\mathbf{g}} \phi_{(j_1 \dots j_k)}\|_{L^2} + \tilde{C} \|e^{-\lambda \tilde{f}_\epsilon} \cdot \tilde{\eta}_\epsilon \xi(\phi_{(j_1 \dots j_k)})\|_{L^2} \\ & + \tilde{C} \left[ \|e^{-\lambda \tilde{f}_\epsilon} \cdot \mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \mathbf{D}^\alpha \tilde{\eta}_\epsilon\|_{L^2} + \|e^{-\lambda \tilde{f}_\epsilon} \cdot \phi_{(j_1 \dots j_k)} (|\square_{\mathbf{g}} \tilde{\eta}_\epsilon| + |D^1 \tilde{\eta}_\epsilon|)\|_{L^2} \right], \end{aligned} \quad (3.53)$$

for any  $\lambda \geq \tilde{C}$ . Using the identities in (1.6), in  $B_{\epsilon^{10}}(x_0)$  we estimate pointwise

$$\begin{cases} |\square_{\mathbf{g}} \phi_{(j_1 \dots j_k)}| \leq \tilde{C}_{\mathcal{A}, \mathcal{B}, \mathcal{C}} \sum_{l_1, \dots, l_k} (|D^1 \phi_{(l_1 \dots l_k)}| + |\phi_{(l_1 \dots l_k)}|); \\ |\xi(\phi_{(j_1 \dots j_k)})| \leq \tilde{C}_{\mathcal{A}, \mathcal{B}, \mathcal{C}} \sum_{l_1, \dots, l_k} |\phi_{(l_1 \dots l_k)}|, \end{cases} \quad (3.54)$$

for some constant  $\tilde{C}_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$  that depends only on the constants  $\tilde{C}$  and the tensors  $\mathcal{A}, \mathcal{B}, \mathcal{C}$ . We add up the inequalities (3.53) over  $(j_1, \dots, j_k) \in \{1, 2, 3, 4\}^k$ . The key observation is that, in view of (3.54), the first two terms in the right-hand side can be absorbed into the left-hand side for  $\lambda$  sufficiently large. Thus, for any  $\lambda \geq \tilde{C}_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$

$$\begin{aligned} & \lambda \sum_{j_1, \dots, j_k} \|e^{-\lambda \tilde{f}_\epsilon} \cdot \tilde{\eta}_\epsilon \phi_{(j_1 \dots j_k)}\|_{L^2} \\ & \leq \tilde{C} \sum_{j_1, \dots, j_k} \left[ \|e^{-\lambda \tilde{f}_\epsilon} \cdot \mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \mathbf{D}^\alpha \tilde{\eta}_\epsilon\|_{L^2} + \|e^{-\lambda \tilde{f}_\epsilon} \cdot \phi_{(j_1 \dots j_k)} (|\square_{\mathbf{g}} \tilde{\eta}_\epsilon| + |D^1 \tilde{\eta}_\epsilon|)\|_{L^2} \right]. \end{aligned} \quad (3.55)$$

Using the hypothesis (3.48) and the definition of the function  $\tilde{\eta}_\epsilon$ , we have

$$|\mathbf{D}_\alpha \phi_{(j_1 \dots j_k)} \mathbf{D}^\alpha \tilde{\eta}_\epsilon| + \phi_{(j_1 \dots j_k)} (|\square_{\mathbf{g}} \tilde{\eta}_\epsilon| + |D^1 \tilde{\eta}_\epsilon|) \leq \tilde{C}_\phi \cdot \mathbf{1}_{\{x \in B_{\epsilon^{10}}(x_0) : r \geq R_0 \text{ and } N^{x_0} > \epsilon^{20}/2\}},$$

for some  $\tilde{C}_\phi$  that depends on the smooth functions  $\phi_{j_1 \dots j_k}$ . Using the definition (3.51), we observe also that

$$\inf_{B_{\epsilon^{40}}(x_0)} e^{-\lambda \tilde{f}_\epsilon} \geq e^{-\lambda \ln(\epsilon + \epsilon^{32}/2)} \geq \sup_{\{x \in B_{\epsilon^{10}}(x_0) : r \geq R_0 \text{ and } N^{x_0} > \epsilon^{20}/2\}} e^{-\lambda \tilde{f}_\epsilon}.$$

The identity (3.52) follows by letting  $\lambda \rightarrow \infty$  in (3.55).

The set

$$\{x \in \mathbf{E}^4 : t(x) = 0 \text{ and } r(x) = R_0\}$$

is compact, where  $t : \mathbf{E}^4 \rightarrow \mathbb{R}$  is a smooth function which agrees with coordinate function  $t$  in the Boyer-Lindquist coordinates. It follows from (3.49) that there is  $\epsilon_3 > 0$  such that

$$\mathcal{S} \equiv 0 \text{ in the set } \{x \in \mathbf{E}^4 : t(x) = 0 \text{ and } r(x) < R_0 + \epsilon_3\}. \quad (3.56)$$

We define the vectors  $\tilde{\partial}_1 = \partial_r, \tilde{\partial}_2 = \partial_t, \tilde{\partial}_3 = \partial_\theta, \tilde{\partial}_4 = \partial_\phi \in \mathbb{T}(\tilde{\mathbf{E}}^4)$  and the functions  $\tilde{\phi}_{(j_1 \dots j_k)} = \mathcal{S}(\tilde{\partial}_{j_1}, \dots, \tilde{\partial}_{j_k})$  as in the proof of Lemma 3.5. It follows from the identity (3.47) and (3.56) that

$$\tilde{\phi}_{(j_1 \dots j_k)}(r, t, \theta, \phi) = 0 \text{ if } r < R_0 + \epsilon_3,$$

which completes the proof of the lemma.  $\square$

## 4. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. We define the smooth optical functions  $u, v : \mathcal{E}' \rightarrow (-1/2, \infty)$ ,

$$\begin{cases} u(t, x) = |x| - 1 - t; \\ v(t, x) = |x| - 1 + t, \end{cases} \quad (4.1)$$

where  $\mathcal{E}' = \{(t, x) \in \mathcal{M} : |x| > |t| + 1/2\}$ . Notice that  $\mathcal{E} = \{(t, x) \in \mathcal{E}' : u > 0 \text{ and } v > 0\}$ . For  $R \in [1, \infty)$  we define the relatively compact open set

$$\mathcal{E}_R = \{(t, x) \in \mathcal{E} : (u + 1/2)(v + 1/2) < R\}. \quad (4.2)$$

**Proposition 4.1.** *Assume  $R \geq 1$ . Then there is  $\lambda(R) \gg 1$  such that for any  $\phi \in C_0^2(\mathcal{E}_R)$  and  $\lambda \geq \lambda(R)$*

$$\lambda \cdot \|e^{-\lambda f} \cdot \phi\|_{L^2} + \|e^{-\lambda f} \cdot D\phi\|_{L^2} \leq C_R \lambda^{-1/2} \cdot \|e^{-\lambda f} \cdot \square\phi\|_{L^2}, \quad (4.3)$$

where

$$f = \log(u + 1/2) + \log(v + 1/2) = \log[(|x| - 1/2)^2 - t^2]. \quad (4.4)$$

and  $|D\phi| = (\sum_{\mu=0}^d |\partial_\mu \phi|^2)^{1/2}$ .

The Carleman inequality in Proposition 4.1 suffices to prove Theorem 1.1, by an argument similar to the one given in Lemma 3.4 (which exploits implicitly the bifurcate characteristic geometry of  $\mathcal{H}$ , using a cutoff function of the form  $\eta(uv/\delta)$ , to compensate for the fact that we do not assume vanishing of the derivatives of  $\phi$  on  $\mathcal{H}$ ). Proposition 4.1 can be obtained as a direct consequence of Hörmander's general pseudo-convexity condition (2.1). For the convenience of the reader, we provide below a self-contained elementary proof of Proposition 4.1, in which we verify implicitly a similar pseudo-convexity condition in our simple case and show how it implies the Carleman inequality.

*Proof of Proposition 4.1.* The constants  $C \geq 1$  in this proof may depend on  $R$  and  $d$ . We may assume that  $\phi \in C_0^\infty(\mathcal{E}_R)$  is real-valued. Since all partial derivatives of  $f$  are bounded in  $\mathcal{E}_R$ , for (4.3) it suffices to prove that, for  $\lambda \geq \lambda(R)$ ,

$$\lambda \cdot \|e^{-\lambda f} \cdot \phi\|_{L^2} + \|D(e^{-\lambda f} \cdot \phi)\|_{L^2} \leq C \lambda^{-1/2} \cdot \|e^{-\lambda f} \cdot \square\phi\|_{L^2}. \quad (4.5)$$

To prove estimate (4.5) we start by setting,

$$\phi = e^{\lambda f} \psi \quad (4.6)$$

with  $f = f(u, v)$  as above. Observe that,

$$e^{-\lambda f} \square(e^{\lambda f} \psi) = \square\psi + \lambda(2\mathbf{D}^\beta f \mathbf{D}_\beta \psi + \square f \psi) + \lambda^2(\mathbf{D}^\beta f \mathbf{D}_\beta f) \psi.$$

Thus estimate (4.5) follows from,

$$\lambda \|\psi\|_{L^2} + C^{-1} \|D\psi\|_{L^2} \leq C \lambda^{-1/2} \|L\psi + \lambda(\square f) \psi\|_{L^2},$$

where,

$$\begin{aligned} L\psi &= \square\psi + 2\lambda W\psi + \lambda^2 G\psi, \\ W &= \mathbf{D}^\alpha f \mathbf{D}_\alpha, \quad G = \mathbf{D}^\beta f \mathbf{D}_\beta f. \end{aligned}$$

Since  $\square f$  is bounded on  $\mathcal{E}_R$ , i.e.  $|\square f| \leq C$ , it suffices in fact to show that,

$$\lambda \|\psi\|_{L^2} + C^{-1} \|D\psi\|_{L^2} \leq C\lambda^{-1/2} \|L\psi\|_{L^2} \quad (4.7)$$

We shall establish in fact a lower bound for an integral of the form,

$$E = \langle L\psi, 2\lambda(W - w)\psi \rangle = 2\lambda \int_{\mathcal{E}_R} L\psi (W(\psi) - w\psi) \quad (4.8)$$

where  $w$  is a smooth function on  $\mathcal{E}_R$  we will choose below. In fact we will choose  $w$  such that we can establish the lower bound,

$$E \geq C^{-1} (\lambda \|D\psi\|_{L^2}^2 + \lambda^3 \|\psi\|_{L^2}^2) + \lambda^2 \|(W - w)\psi\|_{L^2}^2 \quad (4.9)$$

Since  $E \leq \|L\psi\|_{L^2}^2 + \lambda^2 \|(W - w)\psi\|_{L^2}^2$  (4.7) easily follows from (4.9).

Now, writing  $L\psi = \square\psi + \lambda^2 G\psi + \lambda(W\psi + w\psi) + \lambda(W\psi - w\psi)$ ,

$$\begin{aligned} E &= 2\lambda \langle L\psi, (W - w)\psi \rangle = 2\lambda^2 \|(W - w)\psi\|_{L^2}^2 + 2\lambda^2 \|W\psi\|_{L^2}^2 - 2\lambda^2 \|w\psi\|_{L^2}^2 + E_1 + E_2 \\ E_1 &= \lambda \langle \square\psi, (2W - 2w)\psi \rangle \\ E_2 &= \lambda^3 \langle G\psi, (2W - 2w)\psi \rangle. \end{aligned} \quad (4.10)$$

Thus, for bounded  $w$  and for  $\lambda$  sufficiently large, (4.9) is an immediate consequence of

$$2\lambda^2 \|W\psi\|_{L^2}^2 + E_1 + E_2 \geq C^{-1} (\lambda \|D\psi\|_{L^2}^2 + \lambda^3 \|\psi\|_{L^2}^2), \quad (4.11)$$

To evaluate  $E_1$  and  $E_2$  we make use of the following simple lemma.

**Lemma 4.2.** *Let  $Q_{\alpha\beta} = \mathbf{D}_\alpha \psi \mathbf{D}_\beta \psi - \frac{1}{2} m_{\alpha\beta} (\mathbf{D}^\mu \psi \mathbf{D}_\mu \psi)$  denote the energy-momentum tensor of the wave operator  $\square = m^{\alpha\beta} \mathbf{D}_\alpha \mathbf{D}_\beta$ . Then,*

$$\begin{aligned} \square\psi \cdot (2W\psi - 2w\psi) &= \mathbf{D}^\alpha (2W^\beta Q_{\alpha\beta} - 2w\psi \cdot D_\alpha \psi + D_\alpha w \cdot \psi^2) \\ &\quad - Q^{\alpha\beta} (\mathbf{D}_\alpha W_\beta + \mathbf{D}_\beta W_\alpha) + 2w \mathbf{D}^\alpha \psi \cdot \mathbf{D}_\alpha \psi - \square_g w \cdot \psi^2, \end{aligned}$$

and

$$G\psi \cdot (2W\psi - 2w\psi) = \mathbf{D}^\alpha (\psi^2 G \cdot W_\alpha) - \psi^2 (2wG + W(G) + G \cdot \mathbf{D}^\alpha W_\alpha).$$

Since  $\psi \in C_0^\infty(\mathcal{E}_R)$  we integrate by parts to conclude that

$$\begin{aligned} E_1 + E_2 &= \lambda \int_{\mathcal{E}_R} 2w \mathbf{D}^\alpha \psi \cdot \mathbf{D}_\alpha \psi - 2\mathbf{D}^\alpha W^\beta \cdot Q_{\alpha\beta} \\ &\quad + \lambda^3 \int_{\mathcal{E}_R} \psi^2 (-2wG - W(G) - G \cdot \mathbf{D}^\alpha W_\alpha) \\ &\quad - \lambda \int_{\mathcal{E}_R} \psi^2 \square_g w. \end{aligned} \quad (4.12)$$

To prove (4.11) we are reduced to prove pointwise bounds for the first two integrands in (4.12). More precisely, dividing by  $\lambda$  and  $\lambda^3$  respectively, it suffices to prove that the pointwise bounds

$$C^{-1}|D\psi|^2 \leq \lambda|W(\psi)|^2 + (w\mathbf{D}^\alpha\psi \cdot \mathbf{D}_\alpha\psi - \mathbf{D}^\alpha W^\beta \cdot Q_{\alpha\beta}), \quad (4.13)$$

and

$$C^{-1} \leq -2wG - W(G) - G \cdot \mathbf{D}^\alpha W_\alpha, \quad (4.14)$$

hold on  $\mathcal{E}_R$ , for  $\lambda$  sufficiently large.

Recall that  $W^\alpha = \mathbf{D}^\alpha f$  and  $G = \mathbf{D}_\alpha f \mathbf{D}^\alpha f$ . Observe that

$$w\mathbf{D}^\alpha\psi \cdot \mathbf{D}_\alpha\psi - \mathbf{D}^\alpha W^\beta \cdot Q_{\alpha\beta} = (\mathbf{D}^\alpha\psi \cdot \mathbf{D}^\beta\psi)[(w + \square f/2)m_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta f]$$

and

$$-2wG - W(G) - G \cdot \mathbf{D}^\alpha W_\alpha = -G(2w + \square f) - 2\mathbf{D}^\alpha f \mathbf{D}^\beta f \cdot \mathbf{D}_\alpha \mathbf{D}_\beta f.$$

Thus, with  $w' = w + \square f/2 \in C^\infty(\mathcal{E}_R)$  (still to be chosen), the inequalities (4.13) and (4.14) are equivalent to the pointwise inequalities

$$C^{-1}|D\psi|^2 \leq \lambda|\mathbf{D}_\alpha f \cdot \mathbf{D}^\alpha\psi|^2 + (\mathbf{D}^\alpha\psi \cdot \mathbf{D}^\beta\psi)(w'm_{\alpha\beta} - \mathbf{D}_\alpha \mathbf{D}_\beta f), \quad (4.15)$$

and

$$C^{-1} \leq -w'(\mathbf{D}_\alpha f \mathbf{D}^\alpha f) - \mathbf{D}^\alpha f \mathbf{D}^\beta f \cdot \mathbf{D}_\alpha \mathbf{D}_\beta f \quad (4.16)$$

on  $\mathcal{E}_R$ , for  $\lambda$  sufficiently large.

Let  $h = e^f$  or, in view of (4.4),  $h = (|x| - 1/2)^2 - t^2$ . In terms of  $h$  making use of the inequality  $h \geq 1/4$  on  $\mathcal{E}_R$ , the inequalities (4.15) and (4.16) are equivalent to

$$C^{-1}|D\psi|^2 \leq \lambda|\mathbf{D}_\alpha h \cdot \mathbf{D}^\alpha\psi|^2 + (\mathbf{D}^\alpha\psi \cdot \mathbf{D}^\beta\psi)(w'm_{\alpha\beta} - h^{-1}\mathbf{D}_\alpha \mathbf{D}_\beta h), \quad (4.17)$$

and

$$C^{-1} \leq \mathbf{D}^\alpha h \mathbf{D}^\beta h (h^{-2}\mathbf{D}_\alpha h \mathbf{D}_\beta h - h^{-1}\mathbf{D}_\alpha \mathbf{D}_\beta h) - w'\mathbf{D}_\alpha h \mathbf{D}^\alpha h, \quad (4.18)$$

provided that  $\lambda$  is sufficiently large. To summarize, we need to find  $w' \in C^\infty(\mathcal{E}_R)$  such that the inequalities (4.17) and (4.18) hold in  $\mathcal{E}_R$ , for all  $\lambda$  sufficiently large.

We shall see below that our function  $h$ , strictly positive and smooth on  $\mathcal{E}_R$  verifies the equation,

$$\mathbf{D}^\alpha h \mathbf{D}_\alpha h = 4h \quad (4.19)$$

We infer by differentiation that,  $\mathbf{D}_\alpha \mathbf{D}_\beta h \mathbf{D}^\beta h = 2\mathbf{D}_\alpha h$  and therefore,

$$\mathbf{D}_\alpha \mathbf{D}_\beta h \mathbf{D}^\alpha h \mathbf{D}^\beta h = 8h.$$

Therefore the right-hand side of (4.18) is equal to  $8 - 4hw'$  and thus inequality (4.18) is equivalent to  $hw' \leq 2 - C^{-1}$  in  $\mathcal{E}_R$ , which is clearly satisfied if

$$w' = h^{-1}(2 - A_0|x|^{-1}) \quad \text{for some constant } A_0 > 0. \quad (4.20)$$

On the other hand, setting  $Y^\alpha = \mathbf{D}^\alpha \psi$  and  $H_{\alpha\beta} = \mathbf{D}_\alpha \mathbf{D}_\beta h$ ,  $\alpha, \beta = 0, \dots, d$  and observing that  $H_{0i} = 0$  for  $i = 1, \dots, d$ , we infer that the right-hand side of (4.17) is equal to,

$$\begin{aligned} E &= \lambda(\mathbf{D}_0 h Y^0)^2 + w'(- (Y^0)^2 + |Y'|^2) - h^{-1}(H_{00}(Y^0)^2 + H_{ij} Y^i Y^j) \\ &= (Y^0)^2(-w' - h^{-1}H_{00}) + |Y'|^2(w' + H_{ij} \hat{Y}^i \hat{Y}^j) + \lambda(\mathbf{D}_0 h Y^0 + \mathbf{D}_i h Y^i)^2 \end{aligned}$$

where  $|Y'|^2 = \sum_{i=1}^d (Y^i)^2$  and  $\hat{Y}^i = |Y'|^{-1} Y^i$ . Since  $h = (|x| - 1/2)^2 - t^2$ , we have  $|h| + |h^{-1}| + |x| + (|x| - 1/2)^{-1} \leq C$  in  $\mathcal{E}_R$ . We compute

$$\mathbf{D}_0 h = -2t, \quad \mathbf{D}_j h = (2 - |x|^{-1})x_j \text{ for } j = 1, \dots, d, \quad (4.21)$$

and

$$\begin{aligned} H_{00} &= \mathbf{D}_0 \mathbf{D}_0 h = -2 \\ H_{ij} &= \mathbf{D}_i \mathbf{D}_j h = (2 - |x|^{-1})\delta_{ij} + x_i x_j |x|^{-3} \text{ for } i, j = 1, \dots, d. \end{aligned} \quad (4.22)$$

Thus we easily check that (4.19) is indeed verified. Setting  $Z = Y \cdot \hat{x}$ , with  $\hat{x}_i = \frac{x_i}{|x|}$ , the expression for  $E$  becomes,

$$\begin{aligned} E &= (Y^0)^2 h^{-1}(2 - hw') + |Y'|^2 h^{-1}(hw' - (2 - |x|^{-1}) - h^{-1}|x|^{-1}Z^2) \\ &\quad + \lambda(-2tY^0 + (2|x| - 1)Z)^2 \\ &= h^{-1}A_0|x|^{-1}(Y^0)^2 + h^{-1}(1 - A_0)|x|^{-1}|Y'|^2 - h^{-1}|x|^{-1}Z^2 + \lambda(-2tY^0 + (2|x| - 1)Z)^2 \end{aligned}$$

To derive the bound,

$$E \geq C^{-1}((Y^0)^2 + |Y'|^2), \quad (4.23)$$

from which (4.17) follows, we rely on the following simple lemma.

**Lemma 4.3.** *Given  $\delta > 0$  there exists  $\lambda$  sufficiently large (depending on  $R$  and  $\delta$ ) such that the following inequality holds:*

$$\begin{aligned} &\lambda \left[ (2|x| - 1)Z - 2tY^0 \right]^2 + h^{-1}A_0|x|^{-1}(Y^0)^2 - h^{-1}|x|^{-1}Z^2 \\ &\geq (Y^0)^2 h^{-1}|x|^{-1} \left( A_0 - \frac{t^2}{(|x| - 1/2)^2} - \delta \right), \end{aligned} \quad (4.24)$$

In view of the lemma the bound (4.23) follows by choosing  $A_0 = 1 - C_0^{-1}$  and  $\delta = C_0^{-1}$ , for  $C_0$  sufficiently large depending on  $R$ . This completes the proof of the proposition.  $\square$

We give below the proof of Lemma 4.3.

*Proof.* Inequality (4.24) is equivalent to,

$$\lambda(2|x| - 1)^2 \left[ Z - \frac{t}{(|x| - 1/2)} Y^0 \right]^2 + h^{-1}|x|^{-1} \left( (Y^0 \frac{t}{|x| - 1/2})^2 - Z^2 \right) + \delta h^{-1}|x|^{-1}(Y^0)^2 \geq 0$$

Setting  $X = \frac{t}{|x|-1/2}Y^0 - Z$  we can rewrite the above inequality in the form,

$$\lambda(2|x| - 1)^2 X^2 + h^{-1}|x|^{-1}X(-X + 2\frac{t}{|x|-1/2}Y^0) + \delta h^{-1}|x|^{-1}(Y^0)^2 \geq 0$$

or, equivalently,

$$X^2(\lambda(2|x| - 1)^2 - h^{-1}|x|^{-1}) + 2\frac{t}{|x|-1/2}XY^0 + \delta h^{-1}|x|^{-1}(Y^0)^2 \geq 0$$

which clearly holds for  $t, x$  in  $\mathcal{E}_R$  and all  $X, Y^0$  in  $\mathbb{R}$  provided that  $\lambda$  is sufficiently large.  $\square$

#### APPENDIX A. EXPLICIT COMPUTATIONS IN THE KERR SPACES

We consider the exterior region  $\mathbf{E}^4$  of the Kerr spacetime of mass  $m$  and angular momentum  $ma$ ,  $a \in [0, m)$ . Following [4, Chapter 6], in the standard Boyer-Lindquist coordinates  $(r, t, \theta, \phi) \in (r_+, \infty) \times \mathbb{R} \times (0, \pi) \times \mathbb{S}^1$ ,  $r_{\pm} = m \pm (m^2 - a^2)^{1/2}$ , the Kerr metric on a dense open subset  $\widetilde{\mathbf{E}}^4$  of  $\mathbf{E}^4$  is

$$ds^2 = -\frac{\rho^2 \Delta}{\Sigma^2}(dt)^2 + \frac{\Sigma^2(\sin \theta)^2}{\rho^2} \left( d\phi - \frac{2amr}{\Sigma^2} dt \right)^2 + \frac{\rho^2}{\Delta}(dr)^2 + \rho^2(d\theta)^2, \quad (\text{A.1})$$

where

$$\begin{cases} \Delta = r^2 + a^2 - 2mr; \\ \rho^2 = r^2 + a^2(\cos \theta)^2; \\ \Sigma^2 = (r^2 + a^2)\rho^2 + 2mra^2(\sin \theta)^2 = (r^2 + a^2)^2 - a^2(\sin \theta)^2\Delta. \end{cases} \quad (\text{A.2})$$

This metric is of the form

$$ds^2 = -e^{2\nu}(dt)^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu_2}(dr)^2 + e^{2\mu_3}(d\theta)^2, \quad (\text{A.3})$$

where

$$\begin{aligned} e^{2\nu} &= \frac{\rho^2 \Delta}{\Sigma^2} \text{ and } \nu = \frac{1}{2}[\ln(\rho^2) + \ln \Delta - \ln(\Sigma^2)] \\ e^{2\psi} &= \frac{\Sigma^2(\sin \theta)^2}{\rho^2} \text{ and } \psi = \frac{1}{2}[\ln(\Sigma^2) + 2\ln(\sin \theta) - \ln(\rho^2)]; \\ \omega &= \frac{2amr}{\Sigma^2}; \\ e^{2\mu_2} &= \frac{\rho^2}{\Delta} \text{ and } \mu_2 = \frac{1}{2}[\ln(\rho^2) - \ln \Delta]; \\ e^{2\mu_3} &= \rho^2 \text{ and } \mu_3 = \frac{1}{2}\ln(\rho^2). \end{aligned} \quad (\text{A.4})$$



We compute

$$\begin{aligned}\partial_r \mu_2 &= \frac{r}{\rho^2} - \frac{r-m}{\Delta} \quad \text{and} \quad \partial_\theta \mu_2 = \frac{-a^2 \sin \theta \cos \theta}{\rho^2}; \\ \partial_r \mu_3 &= \frac{r}{\rho^2} \quad \text{and} \quad \partial_\theta \mu_3 = \frac{-a^2 \sin \theta \cos \theta}{\rho^2};\end{aligned}\tag{A.5}$$

and

$$\begin{aligned}\partial_r \omega &= -\frac{2am}{\Sigma^4} [(3r^2 - a^2)(r^2 + a^2) - a^2(\sin \theta)^2(r^2 - a^2)]; \\ \partial_\theta \omega &= \frac{4a^3mr\Delta \sin \theta \cos \theta}{\Sigma^4}; \\ \partial_r \nu &= \frac{r}{\rho^2} + \frac{r-m}{\Delta} - \frac{2r(r^2 + a^2) - a^2(\sin \theta)^2(r-m)}{\Sigma^2}; \\ \partial_\theta \nu &= a^2 \sin \theta \cos \theta \left( \frac{\Delta}{\Sigma^2} - \frac{1}{\rho^2} \right); \\ \partial_r \psi &= \frac{2r(r^2 + a^2) - a^2(\sin \theta)^2(r-m)}{\Sigma^2} - \frac{r}{\rho^2}; \\ \partial_\theta \psi &= -a^2 \sin \theta \cos \theta \left( \frac{\Delta}{\Sigma^2} - \frac{1}{\rho^2} \right) + \frac{\cos \theta}{\sin \theta}.\end{aligned}\tag{A.6}$$

We fix the frame

$$e_0 = e^{-\nu}(\partial_t + \omega \partial_\phi), \quad e_1 = e^{-\psi} \partial_\phi, \quad e_2 = e^{-\mu_2} \partial_r, \quad e_3 = e^{-\mu_3} \partial_\theta.\tag{A.7}$$

Clearly,  $\mathbf{g}_{\alpha\beta} = (\mathbf{g}^{\alpha\beta}) = \text{diag}(-1, 1, 1, 1)$ , where  $\mathbf{g}_{\alpha\beta} = \mathbf{g}(e_\alpha, e_\beta)$ ,  $\alpha, \beta = 0, 1, 2, 3$ . The dual basis of 1-forms is

$$\eta^0 = e^\nu dt, \quad \eta^1 = e^\psi (d\phi - \omega dt), \quad \eta^2 = e^{\mu_2} dr, \quad \eta^3 = e^{\mu_3} d\theta.\tag{A.8}$$

Also

$$\xi = \partial_t = e^\nu \cdot e_0 - e^\psi \omega \cdot e_1.\tag{A.9}$$

We compute now the covariant derivatives  $\mathbf{D}_{e_i} e_j$ ,  $i, j = 0, 1, 2, 3$ . We use the formula

$$\begin{aligned}\mathbf{g}(Z, \mathbf{D}_Y X) &= \frac{1}{2} (X(\mathbf{g}(Y, Z)) + Y(\mathbf{g}(Z, X)) - Z(\mathbf{g}(X, Y)) \\ &\quad - \mathbf{g}([X, Z], Y) - \mathbf{g}([Y, Z], X) - \mathbf{g}([X, Y], Z)),\end{aligned}\tag{A.10}$$

for any vector fields  $X, Y, Z$ . We have

$$\begin{aligned}
[e_0, e_1] &= 0; \\
[e_0, e_2] &= e^{-\mu_2} \partial_r \nu \cdot e_0 - e^{\psi - \mu_2 - \nu} \partial_r \omega \cdot e_1; \\
[e_0, e_3] &= e^{-\mu_3} \partial_\theta \nu \cdot e_0 - e^{\psi - \mu_3 - \nu} \partial_\theta \omega \cdot e_1; \\
[e_1, e_2] &= e^{-\mu_2} \partial_r \psi \cdot e_1; \\
[e_1, e_3] &= e^{-\mu_3} \partial_\theta \psi \cdot e_1; \\
[e_2, e_3] &= e^{-\mu_3} \partial_\theta \mu_2 \cdot e_2 - e^{-\mu_2} \partial_r \mu_3 \cdot e_3.
\end{aligned} \tag{A.11}$$

With  $[e_i, e_j] = C_{ij}^k e_k$ ,  $C_{ij}^k + C_{ji}^k = 0$ , it follows from (A.10) that

$$\mathbf{D}_{e_j} e_i = -\frac{1}{2} \sum_{k=0}^3 (\mathbf{g}_{jj} \mathbf{g}_{kk} C_{ik}^j + \mathbf{g}_{ii} \mathbf{g}_{kk} C_{jk}^i + C_{ij}^k) e_k. \tag{A.12}$$

Using the table (A.11), this gives

$$\begin{aligned}
\mathbf{D}_{e_0} e_0 &= C_{02}^0 e_2 + C_{03}^0 e_3; \quad \mathbf{D}_{e_1} e_0 = \frac{-1}{2} C_{02}^1 e_2 + \frac{-1}{2} C_{03}^1 e_3; \\
\mathbf{D}_{e_2} e_0 &= \frac{-1}{2} C_{02}^1 e_1; \quad \mathbf{D}_{e_3} e_0 = \frac{-1}{2} C_{03}^1 e_1; \\
\mathbf{D}_{e_0} e_1 &= \frac{-1}{2} C_{02}^1 e_2 + \frac{-1}{2} C_{03}^1 e_3; \quad \mathbf{D}_{e_1} e_1 = (-1) C_{12}^1 e_2 + (-1) C_{13}^1 e_3 \\
\mathbf{D}_{e_2} e_1 &= \frac{-1}{2} C_{02}^1 e_0; \quad \mathbf{D}_{e_3} e_1 = \frac{-1}{2} C_{03}^1 e_0; \\
\mathbf{D}_{e_0} e_2 &= C_{02}^0 e_0 + \frac{1}{2} C_{02}^1 e_1; \quad \mathbf{D}_{e_1} e_2 = \frac{-1}{2} C_{02}^1 e_0 + C_{12}^1 e_1; \\
\mathbf{D}_{e_2} e_2 &= -C_{23}^2 e_3; \quad \mathbf{D}_{e_3} e_2 = -C_{23}^3 e_3; \\
\mathbf{D}_{e_0} e_3 &= C_{03}^0 e_0 + \frac{1}{2} C_{03}^1 e_1; \quad \mathbf{D}_{e_1} e_3 = \frac{-1}{2} C_{03}^1 e_0 + C_{13}^1 e_1; \\
\mathbf{D}_{e_2} e_3 &= C_{23}^2 e_2; \quad \mathbf{D}_{e_3} e_3 = C_{23}^3 e_2.
\end{aligned} \tag{A.13}$$

Let

$$Y = 2r(r^2 + a^2) - a^2(\sin \theta)^2(r - m),$$

and observe that

$$(3r^2 - a^2)(r^2 + a^2) - a^2(\sin \theta)^2(r^2 - a^2) = 2rY - \Sigma^2 > 0, \tag{A.14}$$

and

$$2r\Sigma^2 > \rho^2 Y. \tag{A.15}$$

We compute now the Hessian  $\mathbf{D}^2 r$ . More generally, for a function  $f$  that depends only on  $r$  (i.e.  $e_0(f) = e_1(f) = e_3(f) = 0$ ), using (A.11) and (A.13), and the formula

$$\mathbf{D}_\alpha \mathbf{D}_\beta f = \mathbf{D}_\beta \mathbf{D}_\alpha f = e_\alpha(e_\beta(f)) - \mathbf{D}_{e_\alpha} e_\beta(f),$$

$$\begin{aligned} \mathbf{D}_0 \mathbf{D}_0 f &= -C_{02}^0 e^{-\mu_2} \partial_r f = -\frac{\Delta}{\rho^2} \left( \frac{r}{\rho^2} + \frac{r-m}{\Delta} - \frac{Y}{\Sigma^2} \right) \partial_r f \\ \mathbf{D}_0 \mathbf{D}_1 f &= \frac{1}{2} C_{02}^1 e^{-\mu_2} \partial_r f = \frac{\Delta}{\rho^2} \cdot \frac{ma \sin \theta}{\rho^2 \sqrt{\Delta \Sigma^2}} (2rY - \Sigma^2) \partial_r f \\ \mathbf{D}_1 \mathbf{D}_1 f &= C_{12}^1 e^{-\mu_2} \partial_r f = \frac{\Delta}{\rho^2} \left( \frac{Y}{\Sigma^2} - \frac{r}{\rho^2} \right) \partial_r f \\ \mathbf{D}_2 \mathbf{D}_2 f &= e^{-\mu_2} \partial_r (e^{-\mu_2} \partial_r f) = \frac{\Delta}{\rho^2} \partial_r^2 f - \frac{\Delta}{\rho^2} \left( \frac{r}{\rho^2} - \frac{r-m}{\Delta} \right) \partial_r f \\ \mathbf{D}_2 \mathbf{D}_3 f &= -C_{23}^2 e^{-\mu_2} \partial_r f = \frac{\sqrt{\Delta} a^2 \sin \theta \cos \theta}{\rho^4} \partial_r f \\ \mathbf{D}_3 \mathbf{D}_3 f &= -C_{23}^3 e^{-\mu_2} \partial_r f = \frac{\Delta r}{\rho^4} \partial_r f \\ \mathbf{D}_0 \mathbf{D}_2 f &= \mathbf{D}_0 \mathbf{D}_3 f = \mathbf{D}_1 \mathbf{D}_2 f = \mathbf{D}_1 \mathbf{D}_3 f = 0. \end{aligned} \tag{A.16}$$

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UNIVERSITY OF WISCONSIN – MADISON  
*E-mail address:* `ionescu@math.wisc.edu`

PRINCETON UNIVERSITY  
*E-mail address:* `seri@math.princeton.edu`